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The Weibull Fréchet distribution and its applications

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ABSTRACT

A new four-parameter lifetime model called the Weibull Fréchet distribution is defined and studied. Various of its structural properties including ordinary and incomplete moments, quantile and generating functions, probability weighted moments, Rényi and δ-entropies and order statistics are investigated. The new density function can be expressed as a linear mixture of Fréchet densities. The maximum likelihood method is used to estimate the model parameters. The new distribution is applied to two real data sets to prove empirically its flexibility. It can serve as an alternative model to other lifetime distributions in the existing literature for modeling positive real data in many areas.

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1. Introduction

There are hundreds of continuous univariate distributions. In recent years, several applications from engineering, environmental, financial, biomedical sciences, among other areas, have indicated that data sets following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended distributions, a significant progress has been made towards the generalization of some well-known distributions and their successful applications to problems in these areas.

The Fréchet ('Fr' for short) distribution is one of the important distributions in extreme value theory and it has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, wind speeds and sea waves. For more information about the Fr distribution and its applications, see [9]. Moreover, applications of this distribution in various fields are given in [8], who showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Nadarajah and Kotz [16] discussed the sociological models based on Fr random variables. Zaharim *et al.* [21] applied the Fr model for analyzing the wind speed data. Mubarak [13] studied the Fr progressive type-II censored data with binomial removals.

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Many authors developed generalizations of the Fr distribution. For example, Nadarajah and Kotz [15] pioneered the exponentiated Fr, Nadarajah and Gupta [14] and Barreto-Souza *et al.* [4] studied the beta Fr, Mahmoud and Mandouh [11] proposed the transmuted Fr, Krishna *et al.* [10] introduced the Marshall–Olkin Fr, da Silva *et al.* [19] defined the gamma extended Fr, Elbatal *et al.* [6] studied the transmuted exponentiated Fr, Mead and Abd-Eltawab [12] introduced the Kumaraswamy Fr and Afify *et al.* [2] investigated the transmuted Marshall–Olkin Fr distributions.

The probability density function (pdf) and cumulative distribution function (cdf) of the Fr distribution are given by (for $x > 0$)

$$g(x;\alpha,\beta) = \beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \quad \text{and} \quad G(x;\alpha,\beta) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right], \tag{1}$$

respectively, where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter, respectively.

Let a random variable Z have the Fr distribution (1) with parameters α and β. For $r < \beta$, the rth ordinary and incomplete moments of Z are given by $\mu'_r = \alpha^r\Gamma(1 - r/\beta)$ and $\varphi_r(t) = \alpha^r\gamma(1 - r/\beta, (\alpha/t)^{\beta})$, respectively, where $\Gamma(a) = \int_0^{\infty} y^{a-1}\,\mathrm{e}^{-y}\,\mathrm{d}y$ is the complete gamma function and $\gamma(a,z) = \int_0^{z} y^{a-1}\,\mathrm{e}^{-y}\,\mathrm{d}y$ is the lower incomplete gamma function.

We define and study a new lifetime model called the *Weibull Fréchet* (WFr) distribution. Its main characteristic is that two shape parameters are added in Equation (1) to provide more flexibility for the generated distribution. Based on the Weibull-G family pioneered by Bourguignon *et al.* [5], we construct the four-parameter WFr model and give a comprehensive description of some of its mathematical properties. We aim that it will attract wider applications in engineering, medicine and other areas of research.

Let $g(x;\theta)$ and $G(x;\theta)$ denote the density and cumulative functions of a baseline model with parameter vector θ and consider the Weibull cdf $F(x) = 1 - \mathrm{e}^{-ax^b}$ (for $x > 0$) with positive parameters a and b. Based on this cdf, Bourguignon *et al.* [5] replaced the argument x by $G(x;\theta)/\bar{G}(x;\theta)$, where $\bar{G}(x;\theta) = 1 - G(x;\theta)$, and defined the cdf of the *Weibull-G* family by

$$F(x;a,b,\theta) = ab\int_0^{[G(x;\theta)/\bar{G}(x;\theta)]} t^{b-1}\,\mathrm{e}^{-at^b}\,\mathrm{d}t = 1 - \exp\left\{-a\left[\frac{G(x;\theta)}{\bar{G}(x;\theta)}\right]^b\right\}. \tag{2}$$

An easy interpretation of the above family can be given as follows. Let Y be a lifetime random variable having a continuous cdf $G(x;\theta)$. The odds ratio that a component following the lifetime Y will failure at time x is $G(x;\theta)/\bar{G}(x;\theta)$. Consider that the variability of this odds of failure is represented by the random variable X having the Weibull distribution with scale a and shape b. Then, we have $P(Y \le x) = P[X \le G(x;\theta)/\bar{G}(x;\theta)/] = F(x;a,b,\theta)$, which is just given by Equation (2).

The Weibull-G density function becomes

$$f(x;a,b,\theta) = abg(x;\theta)\left[\frac{G(x;\theta)^{b-1}}{\bar{G}(x;\theta)^{b+1}}\right]\exp\left\{-a\left[\frac{G(x;\theta)}{\bar{G}(x;\theta)}\right]^b\right\}. \tag{3}$$

A random variable X with pdf (3) is denoted by $X \sim \text{Weibull} - \text{G}(a,b,\theta)$. If $b = 1$, it corresponds to the exponential generator.

This paper is unfolded as follows. In Section 2, we define the WFr distribution and provide some plots for its pdf and hazard rate function (hrf). We derive a useful linear representation for its pdf in Section 3. We obtain in Section 4 some mathematical properties of the new distribution including quantile function (qf), ordinary and incomplete moments, mean deviations, probability weighted moments (PWMs), moment generating function (mgf), Rényi and δ-entropies and moments of the residual life and reversed residual life. In Section 5, we obtain the order statistics and their moments. The maximum likelihood estimates (MLEs) of the unknown model parameters and a simulation study are provided in Section 6. In Section 7, we prove empirically the WFr flexibility by means of two real data sets. Finally, in Section 8, we offer some concluding remarks.

2. The WFr distribution

By omitting the dependence on the positive parameters α, β, a and b and substituting Equation (1) in Equation (2), the four-parameter WFr cdf of X is given by (for $x > 0$)

$$F(x) = 1 - \exp\left(-a\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right] - 1\right\}^{-b}\right). \tag{4}$$

The pdf corresponding to Equation (4) is given by

$$f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-b-1}$$
$$\times \exp\left(-a\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right] - 1\right\}^{-b}\right), \tag{5}$$

where α is a scale parameter representing the characteristic life and β, a and b are shape parameters representing the different patterns of the WFr distribution. Henceforth, we denote a random variable X having pdf (5) by $X \sim \text{WFr}(\alpha, \beta, a, b)$. The WFr distribution is a very flexible model that approaches to different distributions when its parameters are changed. It contains the following new special models:

- For $b = 1$, the WFr model reduces to the exponential Fr (ExFr) distribution.
- The WFr model reduces to the Weibull inverse exponential (WIE) model when $\beta = 1$.
- The case $\beta = 2$ refers to the Weibull inverse Rayleigh (WIR) distribution.
- For $b = 1$ and $\beta = 1$, it follows the exponential inverse exponential (ExIE) model.
- For $b = 1$ and $\beta = 2$, we have the exponential inverse Rayleigh (ExIR) distribution.

The reliability function (rf), hrf and cumulative hazard rate function (chrf) of X are, respectively, given by

$$R(x) = \exp\left\{-a\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right] - 1\right\}^{-b}\right\},$$

$$h(x) = ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-b-1}$$

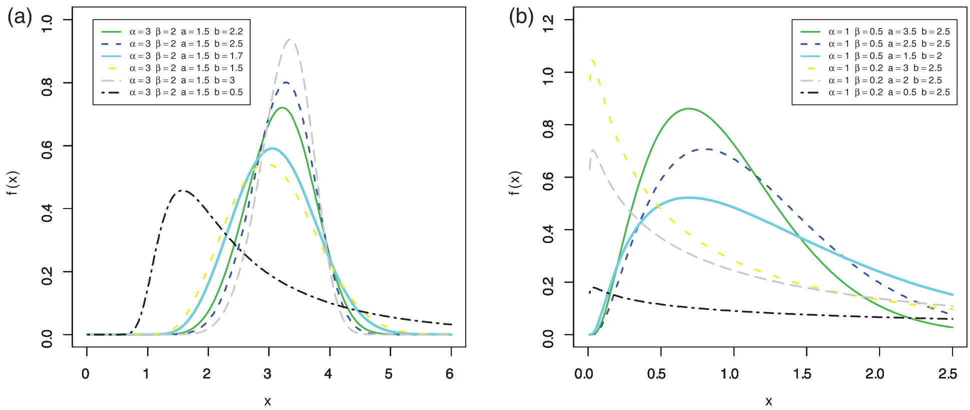

Figure 1. Plots of the WFr pdf for some parameter values [Colour online].

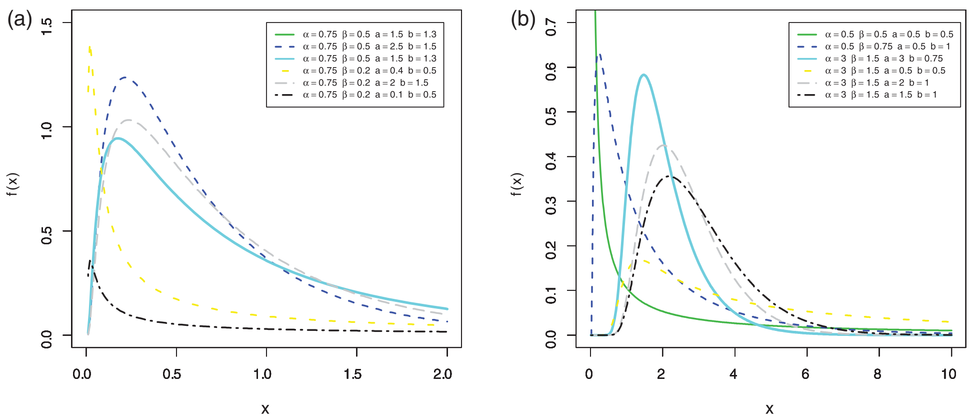

Figure 2. Plots of the WFr pdf for some parameter values [Colour online].

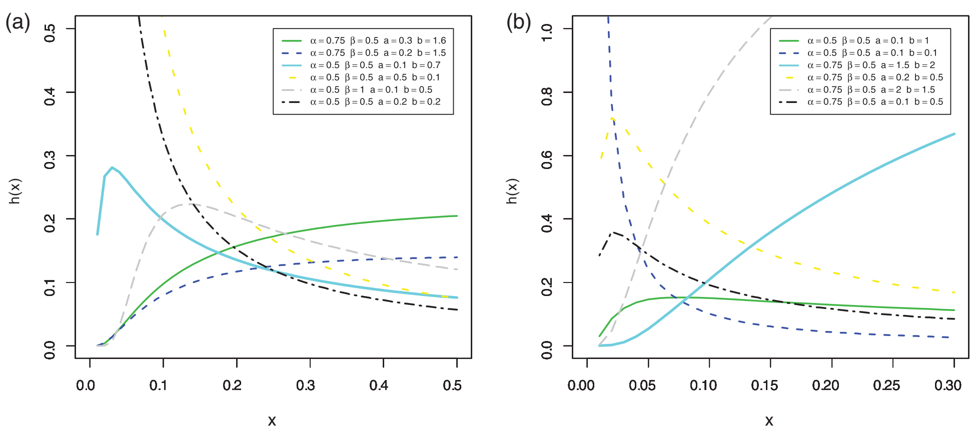

Figure 3. Plots of the WFr hrf for some parameter values [Colour online].

and

$$H(x) = a\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right] - 1\right\}^{-b}.$$

Figures 1 and 2 display some plots of the WFr density for selected values of α, β, a and b. The density plots indicate that the WFr distribution can be skewed to the left and to the right with small and large values for the skewness and kurtosis measures. The plots of the WFr hrf for some parameter values given in Figure 3 reveal that this function can be unimodal, decreasing or increasing, depending on the parameter values.

3. Mixture representation

By substituting Equation (1) in Equation (3), we obtain

$$f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-b-1}$$
$$\times \exp\left(-a\left[\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right]^{b}\right). \quad (6)$$

Let B be the last quantity in Equation (6). By expanding the exponential function in B, we have

$$B = \sum_{k=0}^{\infty}\frac{(-1)^k a^k}{k!}\frac{\exp\left[-kb\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{kb}}.$$

Inserting this expansion in Equation (6) and, after some algebra, we can write

$$f(x) = b\beta\alpha^{\beta}x^{-\beta-1}\sum_{k=0}^{\infty}\frac{(-1)^k a^{k+1}}{k!}\exp\left[-(k+1)b\left(\frac{\alpha}{x}\right)^{\beta}\right]$$
$$\times\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-(kb+b+1)}.$$

By expanding the binomial terms in power series gives

$$f(x) = b\beta\alpha^{\beta}x^{-\beta-1}\sum_{j,k=0}^{\infty}\frac{(-1)^k a^{k+1}[(k+1)b+1]^{(j)}}{j!k!}\exp\left\{-[(k+1)b+j]\left(\frac{\alpha}{x}\right)^{\beta}\right\},$$

where $a^{(j)} = \Gamma(a+j)/\Gamma(a)$ is the rising factorial defined for any real a.

The last equation can be expressed as

$$f(x)=\beta[(k+1)b+j]\alpha^{\beta}\sum_{j,k=0}^{\infty}\upsilon_{j,k}x^{-\beta-1}\exp\left\{-[(k+1)b+j]\left(\frac{\alpha}{x}\right)^{\beta}\right\}, \tag{7}$$

where

$$\upsilon_{j,k}=\frac{(-1)^{k}ba^{k+1}[(k+1)b+1]^{(j)}}{j!k![(k+1)b+j]}.$$

Equation (7) reduces to

$$f(x)=\sum_{j,k=0}^{\infty}\upsilon_{j,k}h_{(k+1)b+j}(x), \tag{8}$$

where $h_{(k+1)b+j}(x)$ is the Fr density with scale parameter $\alpha[(k+1)b+j]^{1/\beta}$ and shape parameter β. Thus, the WFr density can be expressed as a double linear mixture of Fr densities. Then, several of its structural properties can be obtained from Equation (8) and those properties of the Fr distribution.

By integrating Equation (8), the cdf of X can be given in the mixture form

$$F(x)=\sum_{j,k=0}^{\infty}\upsilon_{j,k}H_{(k+1)b+j}(x),$$

where $H_{(k+1)b+j}(x)$ is the Fr cdf with scale parameter $\alpha[(k+1)b+j]^{1/\beta}$ and shape parameter β.

4. Mathematical properties

In this section, we investigate some mathematical properties of the WFr distribution including quantile and random number generation, ordinary and incomplete moments, mean deviations, PWMs, mgf, Rényi and δ-entropies and moments of the residual and reversed residual lifes. Established algebraic expansions to determine some structural properties of the WFr distribution can be more efficient than computing them directly by numerical integration of its density function. Analytical facilities available in programming softwares like `Ox`, `Mathematica`, `Maple`, `R` and `Matlab` can substantially contribute to use these results in practice.

4.1. Quantile and random number generation

For $p\in(0,1)$, the qf of X is obtained by inverting Equation (4) as

$$x_{p}=\alpha[\log\{1+[(-a^{-1})\log(1-p)]^{-1/b}\}]^{-1/\beta},\quad 0<p<1. \tag{9}$$

By setting $p=0.5$ in Equation (9) gives the median M of X. Simulating the WFr random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X=x_p$ at $p=U$ follows Equation (5).

4.2. Moments

The rth ordinary moment of X is given by

$$\mu'_r = E(X^r) = \sum_{j,k=0}^{\infty} \upsilon_{j,k} \int_0^{\infty} x^r h_{(k+1)b+j}(x)\,\mathrm{d}x.$$

For $r < \beta$, we obtain

$$\mu'_r = \sum_{j,k=0}^{\infty} \upsilon_{j,k} \alpha^r [(k+1)b+j]^{r/\beta} \Gamma\left(1-\frac{r}{\beta}\right). \tag{10}$$

Setting $r = 1$ in Equation (10), we have the mean of X.

The nth central moment of X, say μ_n, follows as

$$\mu_n = E(X-\mu)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu_1'^k \mu'_{n-k}.$$

The cumulants (κ_n) of X can be obtained from Equation (10) as

$$\kappa_n = \mu'_n - \sum_{k=0}^{n-1} \binom{n-1}{k-1} \kappa_r \mu'_{n-r},$$

where $\kappa_1 = \mu'_1$. The skewness and kurtosis measures can be evaluated from the ordinary moments using well-known relationships.

4.3. Incomplete moments

The sth incomplete moment, say $\varphi_s(t)$, of the WFr distribution is given by $\varphi_s(t) = \int_0^t x^s f(x)\,\mathrm{d}x$. We can write from Equation (8)

$$\varphi_s(t) = \sum_{j,k=0}^{\infty} \upsilon_{j,k} \int_0^t x^s h_{(k+1)b+j}(x),$$

and then, we obtain (for $s < \beta$),

$$\varphi_s(t) = \alpha^s \sum_{j,k=0}^{\infty} \upsilon_{j,k} [(k+1)b+j]^{s/\beta} \gamma\left(1-\frac{s}{\beta}, [(k+1)b+j]\left(\frac{\alpha}{t}\right)^{\beta}\right).$$

The important application of the first incomplete moment is related to the Bonferroni and Lorenz curves defined by $L(p) = \varphi_1(x_p)/\mu'_1$ and $B(p) = \varphi_1(x_p)/(p\mu'_1)$, respectively, where x_p can be evaluated numerically by Equation (9) for a given probability p. These curves are very useful in economics, demography, insurance, engineering and medicine.

Another application of the first incomplete moment refers to the mean residual life (MRL) and the mean waiting time given by $m_1(t) = [1-\varphi_1(t)]/R(t) - t$ and $M_1(t) = t - \varphi_1(t)/F(t)$, respectively.

Further, the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean and about the median of X (say M) can be expressed as $\delta_\mu = \int_0^\infty |X - \mu_1'| f(x)\,\mathrm{d}x = 2\mu_1' F(\mu_1') - 2\varphi_1 \mu_1'$ and $\delta_M = \int_0^\infty |X - M| f(x)\,\mathrm{d}x = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$ comes from Equation (10), $F(\mu_1')$ is evaluated from Equation (4) and $\varphi_1(\mu_1')$ is the first incomplete moment of X at μ_1'.

4.4. Probability weighted moments

The PWMs can be used to derive estimators of the parameters and quantiles of generalized distributions. These moments have low variances and no severe biases, and they compare favorably with estimators obtained by the maximum likelihood method. The (s, r)th PWM of X (for $r \geq 1,\ s \geq 0$) is formally defined by

$$\rho_{r,s} = E[X^r F(X)^s] = \int_0^\infty x^r F(x)^s f(x)\,\mathrm{d}x.$$

We can write from Equation (4)

$$F(x)^s = \sum_{i=0}^{\infty} (-1)^i \binom{s}{i} \exp\left(-ia\left\{\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}\right\}^b\right).$$

Then, from Equations (4) and (5), we obtain

$$\rho_{r,s} = ab\beta\alpha^\beta \sum_{i=0}^{\infty} (-1)^i \binom{s}{i} \int_0^\infty x^{r-\beta-1} \frac{\exp\left[-b\left(\frac{\alpha}{x}\right)^\beta\right]}{\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]\right\}^{b+1}}$$
$$\times \exp\left(-(i+1)a\left\{\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^\beta\right]}\right\}^b\right)\mathrm{d}x.$$

We can rewrite the last equation as

$$\rho_{r,s} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(k+1+j)} \binom{s}{i} \int_0^\infty x^r h_{(k+1)b+j}(x).$$

By using Equation (10), we obtain (for $r < \beta$)

$$\rho_{r,s} = \sum_{j,i,k=0}^{\infty} d_{j,i,k}\, \alpha^r [(k+1)b+j]^{r/\beta}\, \Gamma\left(1-\frac{r}{\beta}\right),$$

where

$$d_{j,i,k} = \frac{(-1)^{k+i} b a^{k+1} (i+1)^k}{j!k![(k+1)b+j]} [(k+1)b+j+1]^{(j)} [(k+1)b+1]^{r/\beta-1} \binom{s}{i}.$$

4.5. Generating function

First, we obtain the mgf of Equation (1) by setting $y = x^{-1}$

$$M(t;\alpha,\beta) = \beta\alpha^{\beta} \int_0^{\infty} \exp\left(\frac{t}{y}\right) y^{\beta-1} \exp[-(\alpha y)^{\beta}]\,\mathrm{d}y.$$

By expanding the first exponential and determining the integral, we have

$$\begin{aligned} M(t;\alpha,\beta) &= \beta\alpha^{\beta} \int_0^{\infty} \sum_{m=0}^{\infty} \frac{t^m}{m!} y^{\beta-m-1} \exp[-(\alpha y)^{\beta}]\,\mathrm{d}y \\ &= \sum_{m=0}^{\infty} \frac{\alpha^m t^m}{m!} \Gamma\left(\frac{\beta-m}{\beta}\right). \end{aligned}$$

Consider the Wright generalized hypergeometric function defined by

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix};x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)} \frac{x^n}{n!}.$$

Hence, we can write $M(t;\alpha,\beta)$ as

$$M(t;\alpha,\beta) = {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};\alpha t\right]. \tag{11}$$

Combining Equations (8) and (11), the mgf of X, say $M(t)$, reduces to

$$M(t) = \sum_{j,k=0}^{\infty} \upsilon_{j,k}\, {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};\alpha[(k+1)b+j]^{1/\beta}t\right].$$

4.6. Rényi and δ-entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. It is defined by

$$I_{\delta}(X) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f^{\delta}(x)\,\mathrm{d}x, \quad \delta > 0 \quad \text{and} \quad \delta \neq 1.$$

Using Equation (5), we have

$$\begin{aligned} f^{\delta}(x) &= (ab\beta\alpha^{\beta})^{\delta} x^{-\delta(\beta+1)} \exp\left[-b\delta\left(\frac{\alpha}{x}\right)^{\beta}\right] \\ &\times \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-\delta(b+1)} \exp\left\{-a\delta\left[\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right]^{b}\right\}. \end{aligned}$$

After some algebra, we can write

$$f^{\delta}(x) = \sum_{k,j}^{\infty} s_{k,j} x^{-\delta(\beta+1)} \exp\left\{-[(k+\delta)b+j]\left(\frac{\alpha}{x}\right)^{\beta}\right\},$$

where

$$s_{k,j} = (b\beta)^{\delta}\alpha^{\delta\beta}\frac{(-1)^k\delta^k a^{\delta+k}}{k!j!}[(k-\delta)b+\delta]^{(j)}.$$

Then, the Rényi entropy of X reduces to

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\left(\sum_{k,j=0}^{\infty} s_{k,j}\int_0^{\infty} x^{-\delta(\beta+1)}\exp\left\{-[(k+\delta)b+j]\left(\frac{\alpha}{x}\right)^{\beta}\right\}\mathrm{d}x\right).$$

Finally, it can be expressed as

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\left[\Gamma\left(\frac{\delta(\beta+1)}{\beta}\right)\sum_{k,j=0}^{\infty} e_{k,j}\right], \qquad (12)$$

where

$$e_{k,j} = \frac{(-1)^k b^{\delta}\delta^k a^{\delta+k}}{k!j!}\left(\frac{\beta}{\alpha}\right)^{\delta-1}[(k-\delta)b+\delta]^{(j)}[b(k+\delta)+j]^{((1-\delta(\beta+1))/\beta)}.$$

The δ-entropy, say $H_{\delta}(X)$, is defined by

$$H_{\delta}(X) = \frac{1}{\delta-1}\log\left[1-\int_{-\infty}^{\infty} f^{\delta}(x)\,\mathrm{d}x\right], \quad \delta > 0 \quad \text{and} \quad \delta \neq 1,$$

and then it follows from Equation (12).

4.7. Moments of the residual and reversed residual lifes

Several functions are defined related to the residual life, for example, the hrf, MRL function and the left censored mean function. It is well-known that these three functions uniquely determine $F(x)$, see Zoroa *et al.* [22].

Definition 1: *Let X be a random variable representing the life length for a certain unit at age t (where this unit can have multiple interpretations). Then, the random variable $X_t = X - t \mid X > t$ denotes the remaining lifetime beyond that age.*

Further, the nth moment of the residual life of X, namely $m_n(t) = E[(X-t)^n \mid X > t]$ for $n = 1, 2, \ldots$, uniquely determines $F(x)$ (see Navarro *et al.*) [17]. We have

$$m_n(t) = \frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n\,\mathrm{d}F(x).$$

For the WFr distribution, we can write (when $r < \beta$)

$$m_n(t) = \frac{1}{R(t)}\sum_{r=0}^{n}\frac{(-1)^{n-r}n!\alpha^r t^{n-r}}{r!(n-r)!}\sum_{j,k=0}^{\infty}\upsilon_{j,k}[(k+1)b+j]^{r/\beta}$$
$$\times\,\Gamma\left(1-\frac{r}{\beta},[(k+1)b+j]\left(\frac{\alpha}{t}\right)^{\beta}\right),$$

where $\Gamma(a,z) = \int_z^{\infty} y^{a-1}\,\mathrm{e}^{-y}\,\mathrm{d}y$ is the the upper incomplete gamma function.

The MRL function corresponding to $m_n(t)$ represents the expected additional life length for a unit that is alive at age x. Guess and Proschan [7] derived an extensive coverage of possible applications of the MRL function in survival analysis, biomedical sciences, life insurance, maintenance and product quality control, economics, social studies and demography.

In a similar manner, Navarro *et al.* [17] proved that the nth moment of the reversed residual life, say $M_n(t) = E[(t-X)^n \mid X \le t]$ for $t > 0$ and $n = 1, 2, \ldots$, uniquely determines $F(x)$. We obtain

$$M_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n \,\mathrm{d}F(x).$$

For the WFr distribution, we have (when $r < \beta$)

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^{n} \frac{(-1)^r n! \alpha^r}{r!(n-r)!} \sum_{j,k=0}^{\infty} \upsilon_{j,k} [(k+1)b + j]^{r/\beta} \gamma\left(1 - \frac{r}{\beta}, [(k+1)b + j]\left(\frac{\alpha}{t}\right)^{\beta}\right).$$

The mean reversed residual life (MRRL) function corresponding to $M_1(t)$ represents the waiting time elapsed for the failure of an item under the condition that this failure had occurred in $(0, t)$. The MRRL function of X can be obtained by setting $n = 1$ in the above equation .

5. Order statistics

Let $X_1, \ldots, X_n$ be a random sample of size n from the WFr distribution and $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics. Then, the pdf of the ith-order statistic $X_{i:n}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-1}{j} F(x)^{i+j-1}. \tag{13}$$

We can write

$$F(x)^{i+j-1} = \sum_{r=0}^{\infty} (-1)^r \binom{i+j-1}{r} \exp\left(-ra\left\{\frac{\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}^{b}\right), \tag{14}$$

and then by substituting Equations (6) and (14) in Equation (13), we obtain

$$f_{i:n}(x) = \sum_{k,p=0}^{\infty} \upsilon_{k,p} h_{\alpha(bk+b+p)}(x), \tag{15}$$

where

$$\upsilon_{k,p} = \sum_{r=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{j+r+1} b a^{k+1} (r+1)^k [(k+1)b + p + 1]^{(p)}}{k! p! B(i, n-i+1)[(k+1)b + p]} \binom{n-i}{j} \binom{i+j-1}{r}$$

and $h_{\alpha(bk+b+p)}$ denotes the Fr density function with parameters $\alpha[(k+1)b+p]^{1/\beta}$ and β. Thus, the density function of the WFr order statistics is a linear mixture of Fr densities.

Based on Equation (15), we can obtain some structural properties of $X_{i:n}$ from those Fr properties.

For example, the qth moment of $X_{i:n}$ (for $q < \beta$) is given by

$$E(X_{i:n}^q) = \sum_{k,p=0}^{\infty} \upsilon_{k,p} E[Y_{\alpha(bk+b+p)}^q], \tag{16}$$

where $Y_{\alpha(bk+b+p)} \sim \mathrm{Fr}(\alpha[(k+1)b+p]^{1/\beta}, \beta)$.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^d \binom{r-1}{\mathrm{d}} E(X_{r-d:r}), \quad r \geq 1.$$

Based upon the moments in Equation (16), we can obtain explicit expressions for the L-moments of X as infinite weighted linear combinations of suitable WFr means.

6. Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters for this family from complete samples only by maximum likelihood. We investigate the MLEs of the parameters of the WFr(α, β, a, b) model. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample from this model with unknown parameter vector $\theta = (\alpha, \beta, a, b)^\intercal$.

The log-likelihood function for θ, say $\ell = \ell(\theta)$, is given by

$$\ell = n(\log a + \log b + \log\beta + \log\alpha^{\beta}) - (\beta+1)\sum_{i=1}^{n}\log x_i - b\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}$$
$$- (b+1)\sum_{i=1}^{n}(1-s_i) - a\sum_{i=1}^{n}\left(\frac{s_i}{1-s_i}\right)^{b}, \tag{17}$$

where $s_i = \exp[-(\alpha/x_i)^{\beta}]$.

Equation (17) can be maximized either directly by using the `R` (`optim` function), `SAS` (`PROC NLMIXED` sub-routine), `Ox` program (`MaxBFGS`) or by solving the nonlinear likelihood equations obtained by differentiating Equation (17).

The score vector is given by $\mathbf{U}(\theta) = \partial\ell/\partial\theta = (\partial\ell/\partial\alpha, \partial\ell/\partial\beta, \partial\ell/\partial a, \partial\ell/\partial b)^\intercal$.

Let $z_i = (\alpha/x_i)^{\beta}\log(\alpha/x_i)$. Then,

$$\frac{\partial\ell}{\partial\alpha} = \frac{n\beta}{\alpha} - \frac{b\beta}{\alpha^{1-\beta}}\sum_{i=1}^{n}x_i^{-\beta} - \frac{(b+1)\beta}{\alpha^{1-\beta}}\sum_{i=1}^{n}\frac{s_i x_i^{-\beta}}{1-s_i} + \frac{ab\beta}{\alpha^{1-\beta}}\sum_{i=1}^{n}\frac{s_i x_i^{-\beta}}{(s_i-1)^{1-b}},$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + n\log\alpha - \sum_{i=1}^{n}\log x_i - b\sum_{i=1}^{n} z_i - (b+1)\sum_{i=1}^{n}\frac{s_i z_i}{1-s_i} + \sum_{i=1}^{n}\frac{abs_i z_i}{(s_i-1)^{1-b}},$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}(s_i-1)^b$$

and

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log s_i - \sum_{i=1}^{n}\log(1-s_i) - a\sum_{i=1}^{n}(s_i-1)^b\log(s_i-1).$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\hat{\theta}) = \mathbf{0}$. By solving these equations simultaneously gives the MLEs $\hat{\alpha}, \hat{\beta}, \hat{a}$ and $\hat{b}$. These estimates can be obtained numerically using iterative techniques such as the Newton–Raphson algorithm. For the WFr distribution, all the second-order derivatives exist.

For interval estimation of the model parameters, we require the 4×4 observed information matrix $J(\theta) = \{J_{rs}\}$ for $r, s = \alpha, \beta, a, b$. Under standard regularity conditions, the multivariate normal $N_4(0, J(\hat{\theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\hat{\theta})$ is the total observed information matrix evaluated at $\hat{\theta}$. Then, approximate $100(1-\phi)\%$ confidence intervals for the model parameters can be determined in the usual way of the first-order asymptotic theory.

6.1. Simulations study

Various simulations are considered for different sample sizes to examine the performance of the MLEs for the WFr parameters. The simulations are performed as follow:

- The data are generated from $x = \alpha\{\log[(-a)^{1/b}[\log(1-u)]^{-1/b} + 1]\}^{-1/\beta}$, where $u \sim$ U(0, 1).
- The parameter values are set at $\alpha = 2.0$, $\beta = 3.0$, $a = 1.5$ and $b = 0.5$.
- The sample sizes are taken as $n = 50$, 150 and 300.
- Each sample size is replicated 1000 times.

We evaluate the average estimates (AEs), biases and means squared errors (MSEs). The results of the Monte Carlo study are given in Table 1. The figures in this table indicate that the MSEs of the MLEs of the parameters decay toward zero as the sample size increases, as usually expected under first-order asymptotic theory. As the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values. This fact indicates that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the estimates. The usual normal approximation can be oftentimes improved by making bias adjustments to the estimates. Approximations to the biases of the MLEs in simple models may be determined analytically. First-order bias correction typically does a very good job in reducing the bias. However, it may increase the MSE. Whether bias correction is useful in practice depends basically on the shape of the bias function and on the variance of the MLE. In order to improve the accuracy of

Table 1. Simulation results: mean estimates, biases and MSEs of $\hat{\alpha}$, $\hat{\beta}$, $\hat{a}$ and $\hat{b}$.

n	Parameter	AE	Bias	MSE
50	α	1.9910	−0.009	0.0230
	β	3.0846	0.0846	0.2283
	a	1.5188	0.0188	0.0735
	b	0.5208	0.0208	0.0181
150	α	2.0103	0.0103	0.0090
	β	3.0006	0.0001	0.0637
	a	1.5329	0.0329	0.0338
	b	0.5067	0.0067	0.0053
300	α	2.0112	0.0112	0.0040
	β	3.0068	0.0068	0.0274
	a	1.5250	0.0250	0.0141
	b	0.4986	−0.0014	0.0024

these estimates using analytical bias reduction, one needs to obtain several cumulants of log likelihood derivatives, which are notoriously cumbersome for the proposed model.

7. Data analysis

In this section, we prove empirically the flexibility of the new distribution by means of two real data sets. We compare the fits of the WFr, Kumaraswamy Fr (KFr) [12], exponentiated Fr (EFr) [15], beta Fr (BFr) [14], gamma extended Fr (GEFr) [19], transmuted Marshall–Olkin Fr (TMOFr) [2], transmuted Fr (TFr) [11], Marshall–Olkin Fr (MOFr) [10] and Fr distributions. Their density functions (for $x > 0$) are given by:

$$\text{KFr}: f(x;\alpha,\beta,a,b) = ab\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

$$\text{EFr}: f(x;\alpha,\beta,a) = a\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1};$$

$$\text{BFr}: f(x;\alpha,\beta,a,b) = \frac{\beta\alpha^{\beta}}{B(a,b)}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

$$\text{GEFr}: f(x;\alpha,\beta,a,b) = \frac{a\beta\alpha^{\beta}}{\Gamma(b)}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1} \times\left(-\log\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right)^{b-1};$$

$$\text{TMOFr}: f(x;\alpha,\beta,a,b) = a\beta\alpha^{\beta}x^{-\beta-1}\left\{a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-2} \times\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left(1+b-2b\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \times\left\{a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-1}\right);$$

$$\text{TFr}: f(x;\alpha,\beta,b) = \beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1+b-2b\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\};$$

$$\text{MOFr}: f(x;\alpha,\beta,a) = a\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-2}.$$

The parameters of the above densities are all positive real numbers except for the TMOFr and TFr distributions for which $|b| \leq 1$.

The first data set consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett [18]. These data have been used by Afify *et al.* [3] to fit the transmuted complementary Weibull geometric distribution.

The second data set [20] consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the measurement units are not given in their paper. These data have also been used by Barreto-Souza *et al.* [4] and Afify and Aryal [1] for fitting the BFr and Kumaraswamy exponentiated Fr (KEFr) distributions, respectively.

Table 2. Some statistics for models fitted to breaking stress of carbon fibres.

	Goodness of fit criteria				
Model	$-2\hat{\ell}$	AIC	BIC	HQIC	CAIC
WFr	**286.6**	**294.6**	**305.0**	**298.8**	**295.0**
EFr	289.7	295.7	303.5	298.9	296.0
KFr	289.1	297.1	307.5	301.3	297.5
BFr	303.1	311.1	321.6	315.4	311.6
GEFr	304	312	332.4	316.2	312.4
TMOFr	302.0	310.0	320.4	314.2	310.4
Fr	344.3	348.3	353.5	350.4	348.4
TFr	344.5	350.5	358.3	353.6	350.7
MOFr	345.3	351.3	359.1	354.5	351.6

Table 3. MLEs and their standard errors (in parentheses) for breaking stress of carbon fibres.

	Estimates			
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
WFr	0.6942	0.6178	0.0947	3.5178
	(0.363)	(0.284)	(0.456)	(2.942)
EFr	69.1489	0.5019	145.3275	
	(57.349)	(0.08)	(122.924)	
KFr	2.0556	0.4654	6.2815	224.18
	(0.071)	(0.00701)	(0.063)	(0.164)
BFr	1.6097	0.4046	22.0143	29.7617
	(2.498)	(0.108)	(21.432)	(17.479)
GEFr	1.3692	0.4776	27.6452	17.4581
	(2.017)	(0.133)	(14.136)	(14.818)
TMOFr	0.6496	3.3313	101.923	0.2936
	(0.068)	(0.206)	(47.625)	(0.27)
Fr	1.8705	1.7766		
	(0.112)	(0.113)		
TFr	1.9315	1.7435		0.0819
	(0.097)	(0.076)		(0.198)
MOFr	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.3091)	

In order to compare the distributions, we consider the following criteria: the $-2\hat{\ell}$ (minus twice the maximized log-likelihood), AIC (Akaike information criterion), CAIC (consistent Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn information criterion). These statistics are given by $\text{AIC} = -2\hat{\ell} + 2k$, $\text{BIC} = -2\hat{\ell} + k\log(n)$, $\text{HQIC} = -2\hat{\ell} + 2k\log[\log(n)]$ and $\text{CAIC} = -2\hat{\ell} + 2kn/(n-k-1)$, where $\hat{\ell}$ denotes the log-likelihood function evaluated at the MLEs, k is the number of model parameters and n is the sample size. The model with lowest values for these statistics could be chosen as the best model to fit the data.

Tables 2 and 4 provide the values of the above statistics for the fitted models to both data sets, whereas the MLEs and their corresponding standard errors (in parentheses) of the model parameters are listed in Tables 3 and 5, respectively. These results are obtained using the `MATHCAD PROGRAM`.

Tables 2 and 4 compare the WFr model with the KFr, EFr, BFr, GEFr, TMOFr, TFr, MOFr and Fr distributions. The WFr model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics (in bold values) among all fitted models to these data. So, it could be chosen as the best model among them. Figure 4 displays the plots of estimated densities of the WFr, KFr, EFr, BFr, GEFr, TMOFr, TFr, MOFr and Fr models, whereas the plots of estimated cdfs

Table 4. Some statistics for models fitted to strengths of 1.5 cm glass fibres.

	Goodness of fit criteria				
Model	$-2\hat{\ell}$	AIC	BIC	HQIC	CAIC
WFr	**31.0**	**39.0**	**47.6**	**42.4**	**39.7**
KFr	39.6	47.6	56.2	51	48.3
EFr	44.3	50.5	56.7	52.8	50.7
TMOFr	48.5	56.5	65.0	59.8	57.1
MOFr	51.1	57.1	63.5	59.6	57.5
BFr	60.6	68.6	77.2	72.0	69.3
GEFr	61.6	69.6	78.1	72.9	70.3
Fr	93.7	97.7	102	99.4	97.9
TFr	94.1	100.1	106.5	102.6	100.5

Table 5. MLEs and their standard errors (in parentheses) for strengths of 1.5 cm glass fibres.

	Estimates			
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
WFr	0.3865	0.2436	1.4762	16.8561
	(0.799)	(0.285)	(4.782)	(20.485)
KFr	2.116	0.740	5.504	857.343
	(4.555)	(0.071)	(7.982)	(153.948)
EFr	7.816	0.999	132.827	
	(2.945)	(0.136)	(116.63)	
TMOFr	0.65	6.8744	376.268	0.1499
	(0.049)	(0.596)	(246.832)	(0.302)
MOFr	0.6812	6.4655	161.6114	
	(0.045)	(0.559)	(91.499)	
BFr	2.0518	0.6466	15.0756	36.9397
	(0.986)	(0.163)	(12.057)	(22.649)
GEFr	1.6625	0.7421	32.112	13.2688
	(0.952)	(0.197)	(17.397)	(9.967)
Fr	1.264	2.888		
	(0.059)	(0.234)		
TFr	1.3068	2.7898		0.1298
	(0.034)	(0.165)		(0.208)

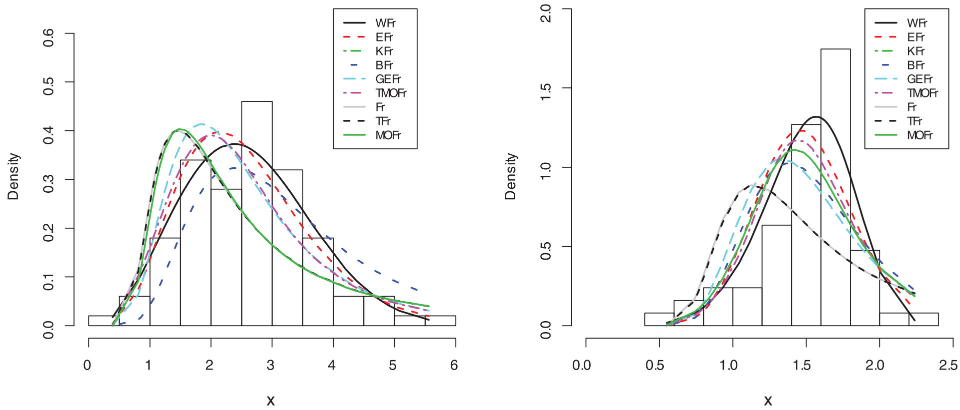

Figure 4. The fitted WFr density and other densities for the first data set (left panel) and second data set (right panel) [Colour online].

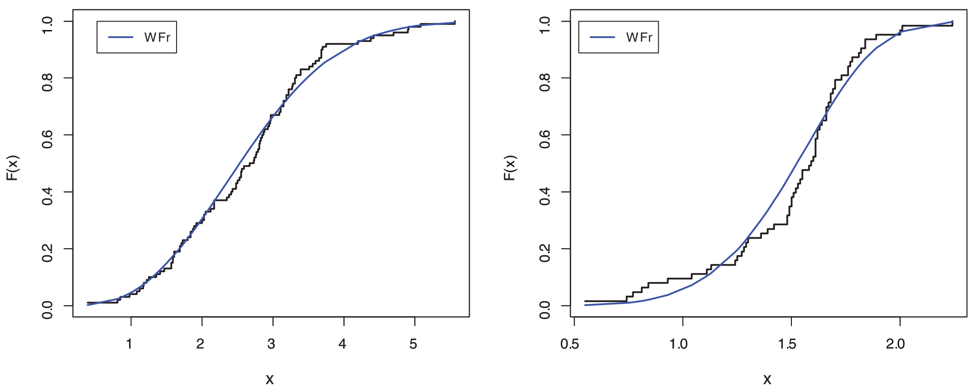

Figure 5. The fitted cdfs of the WFr model for the first data set (left panel) and second data set (right panel) [Colour online].

of the WFr model are displayed in Figure 5. These plots reveal that the WFr distribution yields a better fit than other nested and non-nested models for both data sets.

8. Conclusions

In this paper, we propose a new four-parameter model named the Weibull Fréchet (WFr) distribution, which extends the Fréchet (Fr) distribution. An obvious reason for generalizing a classical distribution is the fact that the generalization provides more flexibility to analyze real life data. We study some of its mathematical and statistical properties. The WFr density function can be expressed as a linear mixture of Fr densities. We derive explicit expressions for the ordinary and incomplete moments, mean deviations, quantile and generating function, PWMs, Rényi and δ-entropies and moments of the residual and reversed residual lifes. We also obtain the density function of the order statistics and their moments. We estimate the model parameters by maximum likelihood. We present a simulation study to illustrate the performance of the estimates. The new distribution applied to two real data sets provides better fits than some other related non-nested models. We hope that the proposed model will attract wider applications in areas such as engineering, survival and lifetime data, meteorology, hydrology, economics (income inequality) and others.

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References

[1] A.Z. Afify and G. Aryal, *The Kumaraswamy exponentiated Fréchet distribution*, J. Data Sci. (2016), forthcoming.
[2] A.Z. Afify, G.G. Hamedani, I. Ghosh, and M.E. Mead, *The transmuted Marshall–Olkin Fréchet distribution: Properties and applications*, Int. J. Statist. Probab. 4 (2015), pp. 132–184.
[3] A.Z. Afify, Z.M. Nofal, and N.S. Butt, *Transmuted complementary Weibull geometric distribution*, Pakistan J. Statist. Oper. Res. 10 (2014), pp. 435–454.
[4] W.M. Barreto-Souza, G.M. Cordeiro, and A.B. Simas, *Some results for beta Fréchet distribution*, Comm. Statist. Theory Methods 40 (2011), pp. 798–811.
[5] M. Bourguignon, R.B. Silva, and G.M. Cordeiro, *The Weibull-G family of probability distributions*, J. Data Sci. 12 (2014), pp. 53–68.
[6] I. Elbatal, G. Asha, and V. Raja, *Transmuted exponentiated Fréchet distribution: Properties and applications*, J. Stat. Appl. Probab. 3 (2014), pp. 379–394.
[7] F. Guess and F. Proschan. *Mean residual life, theory and applications*, in *Handbook of Statistics, 7. Reliability and Quality Control*, P.R. Krishnaiah and C.R., Rao, eds., Elsevier Science Publishers, Amsterdam, 1988, pp. 215–224.
[8] D.G. Harlow, *Applications of the Fréchet distribution function*, Int. J. Mater. Prod. Technol. 17 (2002), pp. 482–495.
[9] S. Kotz and S. Nadarajah, *Extreme Value Distributions: Theory and Applications*, Imperial College Press, London, 2000.
[10] E. Krishna, K.K. Jose, T. Alice, and M.M. Ristic, *The Marshall–Olkin Fréchet distribution*, Comm. Statist. Theory Methods 42 (2013), pp. 4091–4107.
[11] M.R. Mahmoud and R.M. Mandouh, *On the transmuted Fréchet distribution*, J. Appl. Sci. Res. 9 (2013), pp. 5553–5561.
[12] M.E. Mead and A.R. Abd-Eltawab, *A note on Kumaraswamy Fréchet distribution*, Aust. J. Basic Appl. Sci. 8 (2014), pp. 294–300.
[13] M. Mubarak, *Parameter estimation based on the Fréchet progressive type II censored data with binomial removals*, J. Qual. Statist. Reliab. 2012 (2011), 1–5.
[14] S. Nadarajah and A.K. Gupta, *The beta Fréchet distribution*, Far East J. Theor. Stat. 14 (2004), pp. 15–24.
[15] S. Nadarajah and S. Kotz, *The exponentiated Fréchet distribution*, Interstat Electron. J. (2003), pp. 1–7.
[16] S. Nadarajah and S. Kotz, *Sociological models based on Fréchet random variables*, Qual Quant. 42 (2008), pp. 89–95.
[17] J. Navarro, M. Franco, and J. M. Ruiz, *Characterization through moments of the residual life and conditional spacing*, The Indian J. Statist. 60 (1998), Series A, 1, pp. 36–48.
[18] M.D. Nichols and W.J. Padgett, *A bootstrap control chart for Weibull percentiles*, Qual. Reliab. Eng. Int. 22 (2006), pp. 141–151.
[19] R.V. da Silva, T.A.N. de Andrade, D.B.M. Maciel, R.P.S. Campos, and G.M. Cordeiro, *A new lifetime model: The gamma extended Fréchet distribution*, J. Statist. Theory Appl. 12 (2013), pp. 39–54.

[20] R.L. Smith and J.C. Naylor, *A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution*, Appl. Statist. 36 (1987), pp. 358–369.
[21] A. Zaharim, S.K. Najid, A.M. Razali, and K. Sopian, *Analysing Malaysian wind speed data using statistical distribution*, Proceedings of the 4th IASME/WSEAS International Conference on Energy and Environment, Cambridge, 2009.
[22] P. Zoroa, J.M. Ruiz, and J. Marín, *A characterization based on conditional expectations*. Comm. Statist. Theory Methods 19 (1990), pp. 3127–3135.