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The Exponentiated Weibull-H Family of Distributions: Theory and Applications

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Abstract. A new class of continuous distributions called the *exponentiated Weibull-H* family is proposed and studied. The proposed class extends the Weibull-H family of probability distributions introduced by Bourguignon et al. (J Data Sci 12:53–68, 2014). Some special models of the new family are presented. Its basic mathematical properties including explicit expressions for the ordinary and incomplete moments, quantile and generating function, Rényi and Shannon entropies, order statistics, and probability weighted moments are derived. The maximum-likelihood method is adopted to estimate the model parameters and a simulation study is performed. The flexibility of the generated family is proved empirically by means of two applications to real data sets.

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1. Introduction

Determination of a probability distribution which should be adopted to make inference about the data under study is a very important problem in statistics. Because of this, considerable effort over the years has been expended in the development of large classes of distributions along with relevant statistical methodologies. In fact, the statistics literature is filled with hundreds of continuous univariate distributions and their successful applications. There has been a recent renewed interest in generating wider classes of distributions by adding one (or more) shape parameter(s) to a baseline distribution, which makes the generated distribution more flexible, especially for studying tail behavior. Modern computing technology has made many of these techniques accessible even if analytical solutions are very complicated.

In the context of lifetime distributions with cumulative distribution function (cdf) $G(x)$, the most widely used generalization technique is the

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exponentiated-G (exp-G) family. Using this method, for $\nu > 0$, the cdf of the exp-G class is given by the following:

$$F(x;\nu,\boldsymbol{\xi}) = [G(x;\boldsymbol{\xi})]^{\nu}. \tag{1.1}$$

In fact, the method for generating exp-G distributions can be traced back to Lehmann [9]. This generalization technique received a great deal of attention in the last two decades and more than thirty exp-G models have already been published. Some notable examples include the exponentiated Weibull by Mudholkar and Srivastava [12], exponentiated exponential by Gupta et al. [7], exponentiated gamma, exponentiated Fréchet and exponentiated Gumbel by Nadarajah and Kotz [13], exponentiated generalized class of distributions by Cordeiro et al. [5], exponentiated generalized modified Weibull by Aryal et al. [2], and exponentiated Weibull–Pareto by Afify et al. [1], among others. It has been shown that the exp-G models are more flexible and have useful applications in several areas including reliability analysis, biomedical sciences, and environmental studies, among others. Since the exponentiated generalization is more appealing than its classical counterpart, we present the same technique for a class of models studied by Bourguignon et al. [4].

Let $h(x;\boldsymbol{\xi})$, $H(x;\boldsymbol{\xi})$, and $\overline{H}(x;\boldsymbol{\xi})$, respectively, denote the probability density function (pdf), cdf, and reliability function of a baseline model with parameter vector $\boldsymbol{\xi}$ and consider the Weibull pdf $\varphi(x) = abx^{b-1}\exp(-ax^b)$ (for $x > 0$) with parameters $a > 0$ and $b > 0$. Then, the cdf of the *Weibull-H* class is given by the following:

$$G(x;a,b) = \int_0^{\frac{H(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}} \varphi(y)\mathrm{d}y = 1 - \exp\left[-a\left(\frac{H\,(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)^b\right]. \tag{1.2}$$

Hence, the pdf of the *Weibull-H* class reduces to

$$g(x;a,b) = ab\,h(x;\boldsymbol{\xi})\frac{H(x;\boldsymbol{\xi})^{b-1}}{\overline{H}(x;\boldsymbol{\xi})^{b+1}}\exp\left[-a\left(\frac{H(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)^b\right]. \tag{1.3}$$

The rest of the paper is outlined as follows. In Sect. 2, we define the *exponentiated Weibull-H* (EW-H) family of distributions. Five of its special models and some plots of their pdfs and hazard rate functions (hrfs) are presented in Sect. 3. In Sect. 4, we derive a useful linear representation for the EW-H family. In Sect. 5, we obtain some basic mathematical quantities for the new family including ordinary and incomplete moments, mean deviations, quantile function (qf), moment generating function (mgf), and Rényi and q-entropies. Order statistics and their moments are determined in Sect. 6. In Sect. 7, we obtain the probability weighted moments (PWMs). We use maximum likelihood to estimate the model parameters in Sect. 8. In Sect. 9, we perform some simulations to investigate the accuracy and reliability of the maximum-likelihood estimators (MLEs). In Sect. 10, two applications to real data sets prove empirically the flexibility of the new family. Finally, some concluding remarks are offered in Sect. 11.

2. The EW-H family

The proposed family is most conveniently specified in terms of the exponentiated generator applied to the *Weibull-H* class. By inserting (1.2) in equation (1.1), the cdf of the exponentiated Weibull-H (EW-H) family is given by

$$F(x;\nu,a,b,\boldsymbol{\xi}) = \left\{1-\exp\left[-a\left(\frac{H\,(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)^{b}\right]\right\}^{\nu}. \tag{2.1}$$

Therefore, the pdf of the EW-H family reduces to

$$f(x;\nu,a,b,\boldsymbol{\xi}) = ab\nu h(x;\boldsymbol{\xi})\,\frac{H(x;\boldsymbol{\xi})^{b-1}}{\overline{H}(x;\boldsymbol{\xi})^{b+1}}\,\exp\left[-a\left(\frac{H(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)^{b}\right] \times\left\{1-\exp\left[-a\left(\frac{H\,(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)^{b}\right]\right\}^{\nu-1}. \tag{2.2}$$

The additional parameter ν can allow us to study the tail behavior of the density (2.2) with greater flexibility. Furthermore, the EW-H family due to its flexibility in accommodating all forms of the hrf (increasing, decreasing, constant, bathtub, and upside-down bathtub), as shown in Figs. 1b, 2b, 3b, 4b, and 5b, becomes an important family to be used in several applications to real data. A random variable X having pdf (2.2) is denoted by $X \sim$EW-H$(\nu,a,b,\boldsymbol{\xi})$. Some special cases of the EW-H family are listed in Table 1.

In Table 1, the H-model refers to the distribution with cdf $M(x,\boldsymbol{\xi}) = 1-\exp\left(-\frac{H(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)$. This is an exponential distribution of the odds ratio of a continuous random variable whose cdf is given by $H(x;\boldsymbol{\xi})$.

3. Special Models

In this section, we present five special models of the EW-H family. The pdf (2.2) will be most tractable when the cdf $H(x;\boldsymbol{\xi})$ and pdf $h(x;\boldsymbol{\xi})$ have simple analytic expressions. These sub-models generalize several important distributions in the literature. We provide five special models of this family by taking the following baseline distributions: Weibull (W), log-logistic (LL), Fréchet (Fr), Lindely (Li), and Gamma (Ga). The pdfs and cdfs (all defined for $x > 0$) of these baseline models are listed in Table 2. Note that $\Gamma(.)$ and

Table 1. Sub-models of the EW-H family

Reduced model	ν	a	b	Author
W-H family	1	a	b	Bourguignon et al. [4]
BX-H family	ν	1	2	Yousof et al. [18]
E-H family	1	a	1	–
EE-H family	ν	a	1	Tahir et al. [19]
H-model	1	1	1	–

Table 2. Pdf and cdf of baseline models of EW-H family

Model	pdf: $h(x;\boldsymbol{\xi})$	cdf: $H(x;\boldsymbol{\xi})$
W	$\beta\alpha^{\beta}x^{\beta-1}\,\exp[-(\alpha x)^{\beta}]$	$1-\exp[-(\alpha x)^{\beta}]$
LL	$\beta\alpha^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}$	$1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}$
Fr	$\beta\alpha^{\beta}x^{-(\beta+1)}\,\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$	$\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$
Li	$\frac{\alpha^2}{1+\alpha}(1+x)\exp(-\alpha x)$	$1-\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)$
Ga	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x}{\beta}\right)$

$\gamma(.,.)$ in Table 2 denote the gamma function and incomplete gamma function, respectively.

3.1. The EWW Distribution

The cdf and pdf of the EWW distribution are given, respectively, by

$$F(x) = (1-\exp\{-a[\exp(\alpha x)^{\beta}-1]^{b}\})^{\nu}$$

and

$$f(x) = ab\nu\beta\alpha^{\beta}x^{\beta-1}\exp\{b(\alpha x)^{\beta}-a[\exp(\alpha x)^{\beta}-1]^{b}\}$$
$$\times[1-\exp\{-(\alpha x)^{\beta}\}]^{b-1}\,(1-\exp\{-a[\exp(\alpha x)^{\beta}-1]^{b}\})^{\nu-1}.$$

For $\nu = 1$, the EWW distribution reduces to the Weibull-W (WW) distribution. For $\beta = 1$ and $\beta = 2$, we obtain the EW-exponential (EWE) and EW-Rayleigh (EWR) distributions, respectively. Some plots of the pdf and

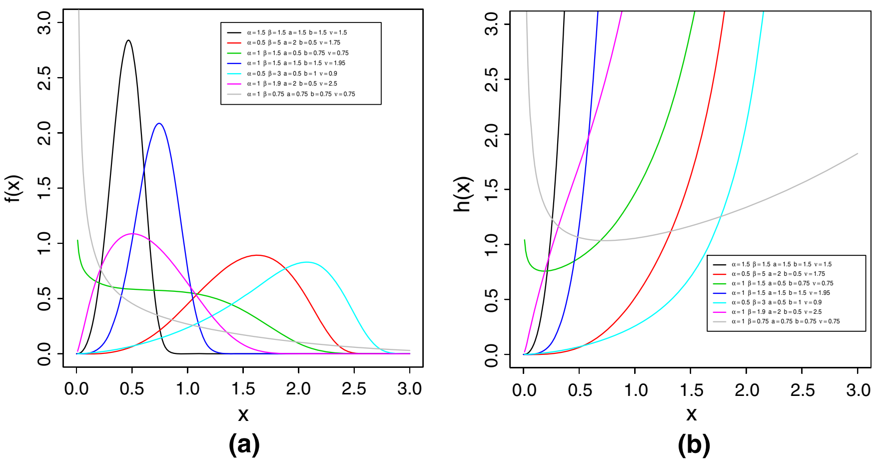

Figure 1. **a** Probability density function of the EWW distribution; **b** hazard rate functions of the EWW distribution

hrf of the EWW distribution are displayed in Fig. 1 for selected parameter values.

3.2. The EWLL Distribution

The cdf and pdf of the EWLL distribution are given, respectively, by

$$F(x) = \left(1 - \exp\left[-a\left(\frac{x}{\alpha}\right)^{b\beta}\right]\right)^{\nu}$$

and

$$f(x) = \frac{ab\nu\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{b\beta-1} \exp\left[-a\left(\frac{x}{\alpha}\right)^{b\beta}\right]\left(1 - \exp\left[-a\left(\frac{x}{\alpha}\right)^{b\beta}\right]\right)^{\nu-1}.$$

For $\nu = 1$, the EWLL model reduces to the W-log-logistic (WLL) distribution. For $b = 1$, we obtain the ELL model. Some plots of the pdf and hrf of the EWLL distribution for selected parameter values are displayed in Fig. 2.

3.3. The EWFr Distribution

The cdf and pdf of the EWFr distribution are, respectively, given by

$$F(x) = \left(1 - \exp\left\{-a\left[\exp\left(\frac{\alpha}{x}\right)^{\beta} - 1\right]^{-b}\right\}\right)^{\nu}$$

and

$$f(x) = ab\nu\beta\alpha^{\beta} \frac{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta} - a\left(\exp\left(\frac{\alpha}{x}\right)^{\beta} - 1\right)^{-b}\right]}{x^{\beta+1}\left(\exp\left(\frac{\alpha}{x}\right)^{\beta} - 1\right)^{b+1}}$$
$$\times \left(1 - \exp\left\{-a\left[\exp\left(\frac{\alpha}{x}\right)^{\beta} - 1\right]^{-b}\right\}\right)^{\nu-1}.$$

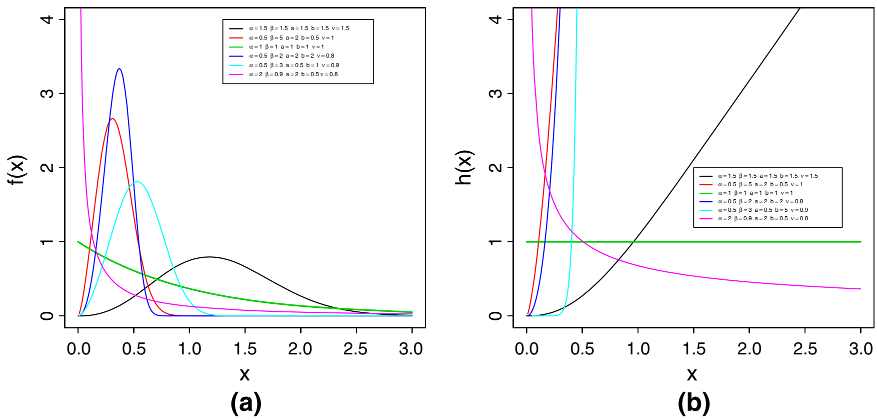

Figure 2. **a** Probability density function of the EWLL distribution; **b** hazard rate functions of the EWLL distribution

For $\nu = 1$, the EWFr distribution becomes the W-Fr distribution. For $\beta = 1$ and $\beta = 2$, we obtain the EW-inverse exponential (EWIE) and EW-inverse Rayleigh (EWIR) distributions, respectively. Plots of the pdf and hrf of the EWFr distribution are displayed in Fig. 3 for some parameter values.

3.4. The EWLi Distribution

The cdf and pdf of the EWLi distribution are given, respectively, by

$$F(x) = \left(1 - \exp\left\{-a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b\right\}\right)^\nu$$

and

$$f(x) = \frac{ab\nu(1+\alpha)\alpha^2(1+x)\exp(\alpha x)}{(1+\alpha+\alpha x)^2}\exp\left\{-a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b\right\}$$
$$\times\left\{\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right\}^{b-1}\left(1 - \exp\left\{-a\left[\frac{(1+\alpha)\exp(\alpha x)}{1+\alpha+\alpha x} - 1\right]^b\right\}\right)^{\nu-1}.$$

For $\nu = 1$, the EWLi model reduces to the WLi distribution. For $b = 1$, we obtain the ELi model. Some plots of the pdf and hrf of the EWLi distribution are displayed in Fig. 4 for some parameter values.

3.5. The EWGa Distribution

The cdf and pdf of the EWGa distribution are given, respectively, by

$$F(x) = \left(1 - \exp\left\{-a\left[\frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma\left(\alpha, \frac{x}{\beta}\right)}\right]^b\right\}\right)^\nu$$

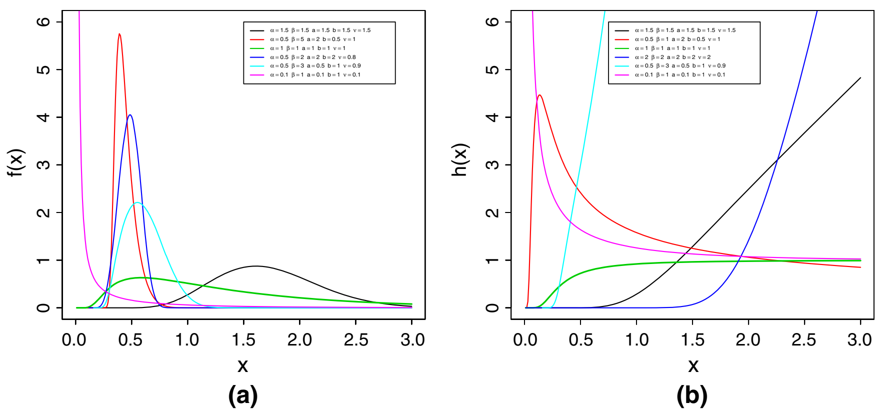

Figure 3. **a** Probability density function of the EWFr distribution; **b** hazard rate functions of the EWFr distribution

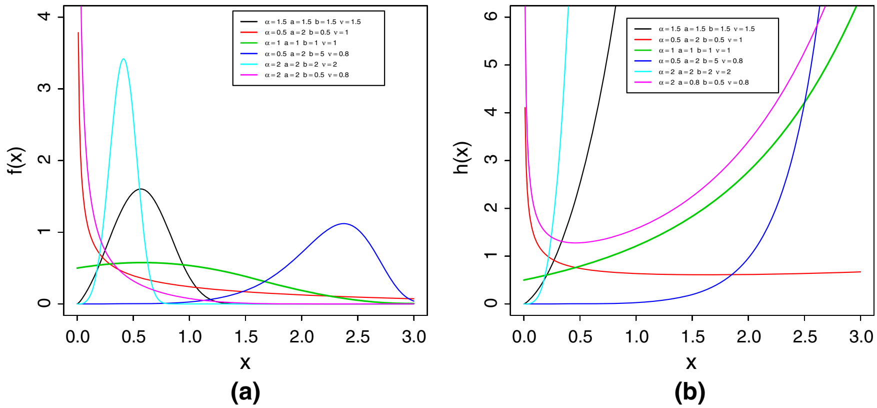

Figure 4. **a** Probability density function of the EWLi distribution; **b** hazard rate functions of the EWLi distribution

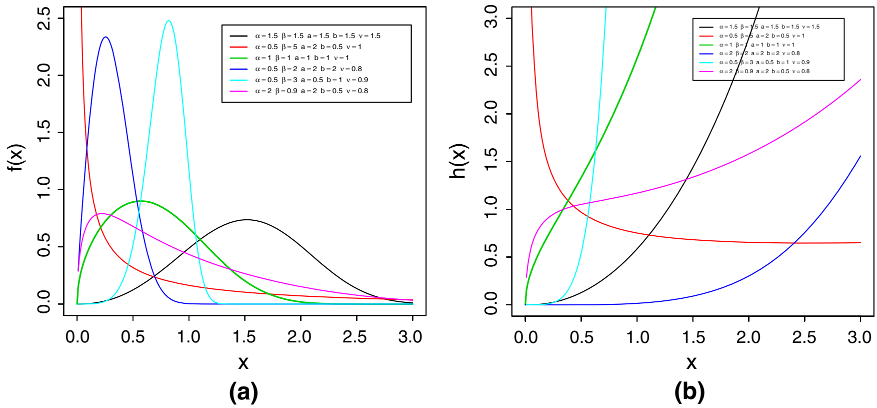

Figure 5. **a** Probability density function of the EWGa distribution; **b** hazard rate functions of the EWGa distribution

and

$$f(x) = \frac{ab\nu\Gamma(\alpha)}{\beta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right) \frac{\left[\gamma\left(\alpha, \frac{x}{\beta}\right)\right]^{b-1}}{\left[\Gamma\left(\alpha, \frac{x}{\beta}\right)\right]^{b+1}}$$
$$\times \exp\left\{-a\left[\frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma\left(\alpha, \frac{x}{\beta}\right)}\right]^{b}\right\}\left(1-\exp\left\{-a\left[\frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma\left(\alpha, \frac{x}{\beta}\right)}\right]^{b}\right\}\right)^{\nu-1}.$$

For $\nu = 1$, the EWGa model reduces to the WGa distribution. Some plots of the pdf and hrf of the EWGa distribution are displayed in Fig. 5 for some parameter values.

4. Linear Representation

In this section, we provide a useful linear representation for the EW-H density function. If $|z| < 1$ and $b > 0$ is a real non-integer, the power series holds

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(b)}{j!\,\Gamma(b-j)}\, z^j. \tag{4.1}$$

Applying (4.1) to the last term in (2.2) gives

$$f(x;\nu,a,b,\boldsymbol{\xi}) = ab\nu\, h(x;\boldsymbol{\xi}) \left(\frac{H(x;\boldsymbol{\xi})^{b-1}}{\overline{H}(x;\boldsymbol{\xi})^{b+1}}\right) \times \sum_{i=0}^{\infty} \frac{(-1)^i\;\Gamma(\nu)}{i!\,\Gamma(\nu-i)} \underbrace{\exp\left[-a\,(i+1)\left(\frac{H(x;\boldsymbol{\xi})}{\overline{H}(x;\boldsymbol{\xi})}\right)^b\right]}_{A_i}. \tag{4.2}$$

Expanding the quantity A_i in power series, we can write

$$A_i = \sum_{k=0}^{\infty} \frac{(-1)^k a^k (i+1)^k}{k!} \frac{H(x;\boldsymbol{\xi})^{kb}}{\overline{H}(x;\boldsymbol{\xi})^{kb}}.$$

Inserting the above expression of A_i in (4.2), the EW-H density reduces to

$$f(x;\nu,a,b,\boldsymbol{\xi}) = h(x;\boldsymbol{\xi}) \sum_{i,k=0}^{\infty} \frac{(-1)^{k+i}\,\nu b a^{k+1}\Gamma(\nu)(i+1)^k}{i!k!\,\Gamma(\nu-i)} \frac{H(x;\boldsymbol{\xi})^{(k+1)b-1}}{\overline{H}(x;\boldsymbol{\xi})^{(k+1)b+1}}. \tag{4.3}$$

Using the generalized binomial expansion to $[1-H(x;\boldsymbol{\xi})]^{-[(k+1)b+1]}$, we can write

$$[1-H(x;\boldsymbol{\xi})]^{-[(k+1)b+1]} = \sum_{j=0}^{\infty} \frac{\Gamma([k+1]b+j+1)}{j!\Gamma([k+1]b+1)} H(x;\boldsymbol{\xi})^j. \tag{4.4}$$

Inserting (4.4) in (4.3), the EW-H density can be expressed as an infinite linear combination of exp-H density functions

$$f(x;\nu,a,b,\boldsymbol{\xi}) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\; \pi_{(k+1)b+j}(x), \tag{4.5}$$

where $\pi_\delta(x) = \delta h(x;\boldsymbol{\xi}) H(x;\boldsymbol{\xi})^{\delta-1}$ is the exp-H pdf with power parameter δ and

$$\upsilon_{k,j} = \sum_{i=0}^{\infty} \frac{(-1)^{k+i}\,\nu b a^{k+1}(i+1)^k\Gamma(\nu)\Gamma([k+1]b+j+1)}{i!\,k!\,j!\,[(k+1)b+j]\Gamma(\nu-i)\Gamma([k+1]b+1)}.$$

Equation (4.5) reveals that the density of X can be expressed as a linear combination of exp-H densities. Therefore, several mathematical properties of the new family can be obtained by knowing those of the exp-H distribution.

Similarly, the cdf of the EW-H family can also be expressed as a linear combination of exp-H cdfs given by

$$F(x;\nu,a,b,\boldsymbol{\xi}) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\; \Pi_{(k+1)b+j}(x),$$

where $\Pi_{(k+1)b+j}(x)$ is the exp-H cdf with power parameter $(k+1)b+j$.

5. Mathematical Properties

The formulae derived in the paper can be handled in most symbolic computation software platforms, for example, Mathematica and Maple because of their ability to deal with complex expressions. Established explicit expressions to determine statistical measures can be more efficient than computing them directly by numerical integration.

5.1. Moments

The rth moment of X, say μ'_r, follows from Eq. (4.5) as

$$\mu'_r = E(X^r) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\, E(Y^r_{(k+1)b+j}),$$

where $Y_{(k+1)b+j}$ denotes the exp-H random variable with power parameter $(k+1)b+j$.

The nth central moment of X, say M_n, is given by

$$\begin{aligned} M_n = E(X-\mu'_1)^n &= \sum_{r=0}^{n} \binom{n}{r} (-\mu'_1)^{n-r}\, E(X^r) \\ &= \sum_{r=0}^{n} \sum_{k,j=0}^{\infty} \binom{n}{r} (-\mu'_1)^{n-r}\, \upsilon_{k,j}\, E(Y^r_{(k+1)b+j}). \end{aligned}$$

5.2. Quantile and Generating Functions

The qf of X is determined by inverting (2.1). We have

$$Q(u) = F^{-1}(u) = H^{-1}\left(\left\{1+\left[-\frac{1}{a}\log\left(1-u^{1/\nu}\right)\right]^{-1/b}\right\}^{-1}\right), \quad 0<u<1.$$

Next, we provide two formulae for the mgf $M_X(t) = E(\mathrm{e}^{\mathrm{t\,X}})$ of X. Clearly, the first one can follow from equation (4.5) as

$$M_X(t) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\, M_{(k+1)b+j}(t),$$

where $M_{(k+1)b+j}(t)$ is the mgf of $Y_{(k+1)b+j}$ (for $k,j \geq 0$). Hence, $M_X(t)$ can be easily obtained from the exp-H generating function.

A second formula for $M_X(t)$ follows from (4.5) as

$$M_X(t) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\, \tau(t,[k+1]b+j+1),$$

where $\tau(t,p) = \int_0^1 \exp[t\,Q_H(u)]\, u^p \mathrm{d}u$ can be evaluated numerically from the baseline qf, i.e., $Q_H(u) = H^{-1}(u)$.

5.3. Incomplete Moments

The main applications of the first incomplete moment are related to the mean deviations and Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance, and medicine. The sth incomplete moment, say $\varphi_s(t)$, of X can be expressed from (4.5) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s\, f(x)\mathrm{d}x = \sum_{k=0}^{\infty} \upsilon_{k,j} \int_{-\infty}^{t} x^s\, \pi_{(k+1)b+j}(x)\mathrm{d}x. \tag{5.1}$$

Clearly, the integral in Eq. (5.1) denotes the sth incomplete moment of $Y_{(k+1)b+j}$.

The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median $[\delta_2 = E(|X - M|)]$ of X are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily evaluated from (2.1), and $\varphi_1\,(t)$ is the first incomplete moment given by (5.1) with $s = 1$.
Now, we provide two ways to determine δ_1 and δ_2. First, a general equation for $\varphi_1(t)$ can be obtained from Eq. (4.5) as

$$\varphi_1\,(t) = \sum_{k=0}^{\infty} \upsilon_{k,j}\, J_{(k+1)b+j}(t),$$

where $J_{(k+1)b+j}(t) = \int_{-\infty}^{t} x\, \pi_{(k+1)b+j}(x)\mathrm{d}x$ is the first incomplete moment of the exp-H distribution.
A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\, \omega_{(k+1)b+j}(t),$$

where $\omega_{(k+1)b+j}(t) = [(k+1)b+j] \int_0^{H(t)} Q_H(u)\, u^{(k+1)b+j}\mathrm{d}u$ can be evaluated numerically. These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined, for a given probability π, by $B(\pi) = \varphi_1(q)/(\pi\mu_1')$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where $q = Q(\pi)$ is the qf of X at π.

5.4. Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. It is defined by

$$I_\theta(X) = \frac{1}{(1-\theta)} \log\left(\int_{-\infty}^{\infty} f(x)^\theta \mathrm{d}x\right), \quad \theta > 0 \text{ and } \theta \neq 1.$$

Using the pdf (2.2), we can obtain after some algebra

$$f(x)^\theta = \sum_{k,j=0}^{\infty} \tau_{k,j}\, h(x)^\theta\, H(x)^{bk+j+(b-1)\theta},$$

where

$$\tau_{k,j} = \sum_{i=0}^{\infty} \frac{(-1)^{i+k} a^{k+\theta} (\nu b)^\theta (\theta+i)^k \Gamma([\nu-1]\theta+1)\Gamma(bk+[b+1]\theta+j)}{i!k!j!\Gamma(bk+[b+1]\theta)\Gamma([\nu-1]\theta-i+1)}.$$

Then, the Rényi entropy of the EW-H family is given by

$$I_\theta(X) = \frac{1}{(1-\theta)} \log \left\{ \sum_{k,j=0}^{\infty} \tau_{k,j} \int_{-\infty}^{\infty} h(x)^\theta H(x)^{bk+j+(b-1)\theta} \mathrm{d}x \right\}.$$

6. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the EW-H family of distributions. The pdf of the ith order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \tag{6.1}$$

where $B(\cdot,\cdot)$ is the beta function. Based on Eqs. (2.1) and (2.2), we have

$$f(x)F^{j+i-1}(x) = \nu ab\, h(x) \frac{H(x)^{b-1}}{\overline{H}(x)^{b+1}} \exp\left[-a\left(\frac{H(x)}{\overline{H}(x)}\right)^b\right] \times \left\{1 - \exp\left[-a\left(\frac{H(x)}{\overline{H}(x)}\right)^b\right]\right\}^{\nu(j+i)-1}.$$

Following the same steps of the linear representation (4.5), we obtain

$$f(x)F^{j+i-1}(x) = h(x) \sum_{l,k=0}^{\infty} \frac{(-1)^{l+k} \nu b a^{k+1} (1+l)^k \Gamma([j+i]\nu)}{l!\, k!\, \Gamma([j+i]\nu - l)} \frac{H(x)^{(k+1)b-1}}{\overline{H}(x)^{(k+1)b+1}}.$$

Then

$$f(x)F^{j+i-1}(x) = \sum_{k,m=0}^{\infty} t_{k,m}^{(j)}\, \pi_{(k+1)b+m}(x), \tag{6.2}$$

where

$$t_{k,m}^{(j)} = \sum_{l=0}^{\infty} \frac{(-1)^{l+k} \nu b\, a^{k+1}\, (1+l)^k \Gamma([j+i]\nu)\, \Gamma([k+1]b+1+m)}{l!\, k! m!\, [(k+1)b+m]\, \Gamma([j+i]\nu - l)\, \Gamma([k+1]b+1)}.$$

Substituting (6.2) in Eq. (6.1), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \frac{1}{\mathrm{B}(i, n-i+1)} \sum_{k,m=0}^{\infty} q_{k,m}\, \pi_{(k+1)b+m}(x),$$

where $\pi_{(k+1)b+m}(x)$ is the exp-H density with power parameter $(k+1)b+m$ and $q_{k,m} = \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} t_{k,m}^{(j)}$. The density function of the EW-H order statistics is a linear combination of exp-H densities. Based on the last equation, we note that the main properties of $X_{i:n}$ follow from those properties of Y_{k+1}. For example, the moments of $X_{i:n}$ are given by

$$E(X_{i:n}^s) = \frac{1}{\mathrm{B}(i, n-i+1)} \sum_{k,m=0}^{\infty} q_{k,m}\, E(Y_{(k+1)b+m}^s). \tag{6.3}$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in Eq. (6.3), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable EW-H order statistics.

7. Probability Weighted Moments

The PWMs are expectations of certain functions of a random variable and can be defined for any random variable whose ordinary moments exist. The (s, r)th PWM of the EW-H distribution, say $\rho_{s,r}$, can be formally defined by

$$\rho_{s,r} = E\{X^s\, F(X)^r\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\mathrm{d}x.$$

From Eqs. (2.1) and (2.2), we can write

$$f(x)\, F(X)^r = \sum_{k,j=0}^{\infty} d_{k,j}^{(r)}\, \pi_{(k+1)b+j}(x),$$

where

$$d_{k,j}^{(r)} = \sum_{i=0}^{\infty} \frac{(-1)^{k+i} \nu\, b\, a^{k+1} (i+1)^k \Gamma([r+1]\nu)\Gamma([k+1]b+j+1)}{i!k!\,j!\,[(k+1)b+j]\Gamma([r+1]\nu-i)\Gamma([k+1]b+1)}.$$

Then, $\rho_{s,r}$ can be expressed as

$$\rho_{s,r} = \sum_{k,j=0}^{\infty} d_{k,j}^{(r)} \int_{-\infty}^{\infty} x^s\, \pi_{(k+1)b+j},(x)\mathrm{d}x.$$

Finally, the (s, r)th PWM of X can be obtained from an infinite linear combination of exp-H moments given by

$$\rho_{s,r} = \sum_{k,j=0}^{\infty} d_{k,j}^{(r)}\, E(Y_{(k+1)b+j}^s).$$

8. Parameter Estimation

Several approaches for parameter estimation were proposed in the literature, but the maximum-likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used for constructing confidence intervals for the model parameters and also for hypothesis testing. Therefore, we consider the estimation of the unknown parameters for this family from complete samples only by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from the EW-H family with parameters ν, a, b and $\boldsymbol{\xi}$. Let $\boldsymbol{\theta}$ $=(\nu, a, b, \boldsymbol{\xi}^\intercal)^\intercal$ be the $(p \times 1)$ parameter vector. The standard regularity conditions are satisfied for all distributions other than the one whose support depends on an unknown parameter. The log-likelihood function ℓ for the EW-H distribution is given by

$$\ell = n(\log\nu + \log a + \log b) + \sum_{i=1}^{n} \log h(x_i;\boldsymbol{\xi}) + (b-1)\sum_{i=1}^{n} \log H(x_i;\boldsymbol{\xi})$$
$$-(b+1)\sum_{i=1}^{n} \log \overline{H}(x_i;\boldsymbol{\xi}) - a\sum_{i=1}^{n} s_i^b + (\nu-1)\sum_{i=1}^{n} \log q_i, \tag{8.1}$$

where $s_i = H(x_i;\boldsymbol{\xi})/\overline{H}(x_i;\boldsymbol{\xi})$ and $q_i = 1 - \exp(-as_i^b)$ (for $i = 1,\ldots,n$). Equation (8.1) can be maximized either directly using the `R` (`optim` function), `SAS` (`PROC NLMIXED`), `Ox` program (`MaxBFGS` sub-routine), and `MATH-CAD` program or by solving the nonlinear likelihood equations obtained by differentiating (8.1).

The score vector components, say $\boldsymbol{U}(\boldsymbol{\theta}) = \frac{\partial\ell}{\partial\boldsymbol{\theta}} = (\frac{\partial\ell}{\partial\nu}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\,\boldsymbol{\xi}})^{\mathsf{T}}$, are given by

$$U_\nu = \frac{n}{\nu} + \sum_{i=1}^{n} \log[1 - \exp(-as_i^b)],$$
$$U_a = \frac{n}{a} - \sum_{i=1}^{n} s_i^b + (\nu-1)\sum_{i=1}^{n} \frac{\exp(-as_i^b)s_i^b}{q_i},$$
$$U_b = \frac{n}{b} + \sum_{i=1}^{n} \log H(x_i;\boldsymbol{\xi}) - \sum_{i=1}^{n} \log \overline{H}(x_i;\boldsymbol{\xi})$$
$$-a\sum_{i=1}^{n} s_i^b \log s_i + a(\nu-1)\sum_{i=1}^{n} \frac{\exp(-as_i^b)s_i^b \log s_i}{q_i}$$

and

$$U_{\xi_k} = \sum_{i=1}^{n} \frac{h'_{\xi_k}(x_i;\boldsymbol{\xi})}{h(x_i;\boldsymbol{\xi})} + (b-1)\sum_{i=1}^{n} \frac{H'_{\xi_k}(x_i;\boldsymbol{\xi})}{H(x_i;\boldsymbol{\xi})} - (b+1)\sum_{i=1}^{n} \frac{\overline{H}'_{\xi_k}(x_i;\boldsymbol{\xi})}{\overline{H}(x_i;\boldsymbol{\xi})}$$
$$-ab\sum_{i=1}^{n} s_i^{b-1}\left(\frac{\partial s_i}{\partial\xi_k}\right) + (\nu-1)\sum_{i=1}^{n} \frac{1}{q_i}\left(\frac{\partial q_i}{\partial\xi_k}\right),$$

where $h'_{\xi_k}(x_i;\boldsymbol{\xi}) = \partial h(x_i;\boldsymbol{\xi})/\partial\xi_k$, $H'_{\xi_k}(x_i;\boldsymbol{\xi}) = \partial H(x_i;\boldsymbol{\xi})/\partial\xi_k$ and $\overline{H}'_{\xi_k}(x_i;\boldsymbol{\xi}) = \partial\overline{H}(x_i;\boldsymbol{\xi})/\partial\xi_k$.

Setting the non-linear system of equations $U_\nu = U_a = U_b = U_{\xi_k} = 0$ and solving them simultaneously yield the MLE $\widehat{\boldsymbol{\theta}} = (\widehat{\nu}, \widehat{a}, \widehat{b}, \widehat{\boldsymbol{\xi}}^{\mathsf{T}})^{\mathsf{T}}$ of $\boldsymbol{\theta}$. These equations can be solved numerically using iterative methods such as the Newton–Raphson type algorithms. For interval estimation of the model parameters, we require the observed information matrix $J(\boldsymbol{\theta})$, which can be obtained from the authors upon request. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\boldsymbol{\theta}}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\boldsymbol{\theta}})$ is the total observed information

Table 3. Empirical means and the RMSEs of the EWLL distribution for $a = 1$, $b = 2$, $\alpha = 2$, $\beta = 1$, and $\nu = 2$

n	$\widehat{a}$	$\widehat{b}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\nu}$
100	1.08803	2.01018	2.04639	1.04040	2.54616
	(0.36499)	(0.15986)	(0.27339)	(0.24770)	(2.14516)
200	1.05870	1.99945	2.02488	1.01272	2.24989
	(0.21298)	(0.09229)	(0.17682)	(0.16108)	(0.99694)
300	1.04556	1.99769	2.02496	1.01177	2.15514
	(0.18393)	(0.07653)	(0.14252)	(0.13042)	(0.77584)
400	1.03653	1.99893	2.02017	1.00688	2.11859
	(0.14742)	(0.06459)	(0.13414)	(0.11176)	(0.64036)
500	1.03846	1.99710	2.02366	1.00582	2.09826
	(0.14121)	(0.06329)	(0.12053)	(0.10235)	(0.59191)
1000	1.02154	1.99922	2.01458	1.00254	2.04608
	(0.09039)	(0.03549)	(0.07877)	(0.06788)	(0.35141)

matrix evaluated at $\widehat{\boldsymbol{\theta}}$. Large sample theory for these estimators delivers simple approximations that work well in finite samples. The normal approximation for the MLEs is easily handled numerically. Likelihood ratio tests can be performed for the proposed family in the usual way.

9. Simulation Study

In this section, we present some simulations for different sample sizes to assess the accuracy of the MLEs. For illustrative purposes, we will choose the EWLL distribution. An ideal technique for simulating from the EWLL distribution is the inversion method. We can simulate X by

$$X = \alpha \left[-\frac{1}{a} \log(1 - U^{1/\nu}) \right]^{1/(b\beta)},$$

where U is a uniform random number in $(0, 1)$. For selected combinations of a, b, α, β and ν, we generate samples of sizes $n = 100, 200, 300, 400, 500$, and $1,000$ from the EWLL distribution. We repeat the simulations $N = 1,000$ times and evaluate the mean estimates and the root-mean-square errors (RMSEs). The empirical results obtained using the statistical computing software R are given in Tables 3 and 4. It can be noted that as sample size increases, the mean square error decreases. Therefore, the maximum-likelihood method works very well to estimate the model parameters of the EWLL distribution.

10. Applications

In this section, we prove empirically the flexibility of the EWLi and EWFr models presented in Sect. 3 by means of two applications to real data. The

Table 4. Empirical means and the RMSEs of the EWLL distribution for $a = 2$, $b = 1$, $\alpha = 2$, $\beta = 2$ and $\nu = 1$

n	$\widehat{a}$	$\widehat{b}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\nu}$
100	2.05735	1.07065	1.99478	2.02222	1.14787
	(0.18058)	(0.27945)	(0.20209)	(0.18293)	(0.88286)
200	2.03073	1.02989	2.00458	2.01052	1.04472
	(0.11937)	(0.16233)	(0.12749)	(0.09871)	(0.35811)
300	2.02832	1.02531	2.01255	2.00729	1.01846
	(0.10232)	(0.12902)	(0.10713)	(0.07846)	(0.28124)
400	2.03168	1.01404	2.00601	2.00005	1.02899
	(0.10686)	(0.10969)	(0.08896)	(0.06778)	(0.24692)
500	2.02512	1.01295	2.00914	2.00229	1.01726
	(0.08558)	(0.09685)	(0.08227)	(0.05829)	(0.21188)
1000	2.02019	1.00494	2.00554	2.00007	1.01158
	(0.06810)	(0.06733)	(0.05644)	(0.03805)	(0.14809)

MLEs of the model parameters and some goodness-of-fit statistics for the fitted models are computed using MATH-CAD.

10.1. The Cancer Patient Data

The first data set refers to the remission times (in months) of a random sample of 128 bladder cancer patients provided in Lee and Wang [8]. We compare the fits to these data of the EWLi, WLi, EELi, extended Lindley (ELi)(Bakouch et al. [3]), power Lindley (PLi) (Ghitnay et al. [6]), and Lindley (Li) distributions. Note that the pdfs of the ELi and PLi distributions are, respectively, given by

$$\text{ELi} : f(x) = \frac{\alpha(1+\alpha+\alpha x)^{\nu-1}}{(1+\alpha)^{\nu}} \exp[-(\alpha x)^{\beta}][\beta(1+\alpha+\alpha x)(\alpha x)^{\beta-1} - \nu];$$

$$\text{PLi} : f(x) = \frac{\beta\alpha^2}{(1+\alpha)} x^{\beta-1}(1+x^{\beta}) \exp(-\alpha x^{\beta}).$$

The parameters of the above densities are all positive real numbers except $\nu \in \mathbb{R}^{-} \cup \{0, 1\}$ for the ELi model.

10.2. Breaking Stress of Carbon Fibre Data

The second data set consists of 100 observations from Nichols and Padgett [16] on breaking stress of carbon fibre (in Gba). Here, we use these data to compare the fit of the EWFr model with those of the following models: Kumaraswamy Fréchet (KwFr) (Mead and Abd-Eltawab [11]), exponentiated Fréchet (EFr) (Nadarajah and Kotz [15]), beta Fréchet (BFr) (Nadarajah and Gupta [14]), gamma extended Fréchet (GEFr)(Silva et al. [17]), transmuted Fréchet (TFr) (Mahmoud and Mandouh [10]), and Fréchet (Fr) distributions with corresponding densities (for $x > 0$):

$$\text{KwFr}: f(x) = ab\beta\alpha^{\beta}\, x^{-(\beta+1)} \exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

$$\text{EFr}: f(x) = a\beta\alpha^{\beta}\, x^{-(\beta+1)} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\ \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1};$$

$$\text{BFr}: f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}\, x^{-(\beta+1)} \exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\ \left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

$$\begin{aligned}\text{GEFr}: f(x) &= \frac{a\beta\alpha^{\beta}}{\nu(b)}\, x^{-(\beta+1)} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1}\\ &\quad\times\left[-\log\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right]^{b-1};\end{aligned}$$

$$\text{TFr}: f(x) = \beta\alpha^{\beta}x^{-(\beta+1)} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{(a+1)-2a\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}.$$

The parameters of the above densities are all positive real numbers except $|a| \le 1$ for the TFr distribution. To compare the distributions, we consider the goodness-of-fit statistics including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), minus twice maximized log-likelihood under the model $(-2\widehat{\ell})$, Anderson–Darling (A^*), and Cramér–Von Mises (W^*) statistics. They are given by

$$\begin{aligned}\text{AIC} &= -2\widehat{\ell}+2p, \quad \text{BIC} = -2\widehat{\ell}+p\log(n),\\ \text{HQIC} &= -2\widehat{\ell}+2p\log[\log(n)], \quad \text{CAIC} = -2\widehat{\ell}+2pn/(n-p-1),\\ A^* &= \left(\frac{9}{4n^2}+\frac{3}{4n}+1\right)\left\{n+\frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[z_i\left(1-z_{n-j+1}\right)\right]\right\}\end{aligned}$$

and

$$W^* = \left(\frac{1}{2n}+1\right)\left[\sum_{j=1}^{n}\left(z_i-\frac{2j-1}{2n}\right)^2+\frac{1}{12n}\right],$$

respectively, where $z_i = F(y_{(i)})$, p is the number of parameters, n is the sample size, and the values $y_{(i)}$'s are the *ordered observations*. The smaller these statistics are, the better the fit is. Upper tail percentiles of the asymptotic distributions are tabulated in Nichols and Padgett [16]. The MLEs and their corresponding standard errors (in parentheses) of the model parameters for cancer patient data are given in Table 5.

In Table 6, we compare the fits of the EWLi, PLi, ELi, EELi, WLi, and Li distributions. The numerical values in Table 6 indicate that the EWLi model has the lowest values for the $-2\widehat{\ell}$, AIC, CAIC, HQIC, BIC, W^*, and A^* statistics (for cancer data) among the fitted models. Therefore, the EWLi model could be chosen as the best fitted model.

Table 5. MLEs and their standard errors (in parentheses) for cancer patient data

Model	Estimates			
EWLi	$\hat{\alpha} = 0.0161$ (0.026)	$\hat{\nu} = 5.9046$ (3.634)	$\hat{a} = 9.7534$ (8.531)	$\hat{b} = 0.2941$ (0.069)
ELi	$\hat{\alpha} = 0.0444$ (0.059)	$\hat{\nu} = -2.0373$ (4.248)	$\hat{\beta} = 1.2244$ (0.271)	
EELi	$\hat{\alpha} = 0.019$ (0.0052)	$\hat{\nu} = 0.4692$ (0.05)	$\hat{a} = 17.918$ (9.5)	
WLi	$\hat{\alpha} = 0.0199$ (0.009117)	$\hat{a} = 10.6812$ (5.714)	$\hat{b} = 0.6092$ (0.039)	
PLi	$\hat{\alpha} = 0.2944$ (0.037)	$\hat{\beta} = 0.8301$ (0.047)		
Li	$\hat{\alpha} = 0.1961$ (0.012)			

Table 6. Goodness-of-fit statistics for cancer data

Model	$-2\widehat{\ell}$	AIC	CAIC	HQIC	BIC	W^*	A^*
EWLi	819.700	827.700	828.025	832.335	839.108	0.02875	0.18791
PLi	826.697	830.697	830.793	833.015	836.401	0.10248	0.62961
ELi	827.124	833.124	833.317	836.6	841.68	0.0549	0.47177
WLi	834.476	840.476	840.67	843.953	849.032	0.16488	1.03148
Li	839.072	841.072	841.103	842.23	843.924	0.36345	2.26142
EELi	860.315	866.315	866.509	869.792	874.871	0.44481	2.63767

The MLEs and their corresponding standard errors (in parentheses) of the model parameters for breaking stress of carbon fibre data are listed in Table 7.

In Table 8, we provide the fit statistics of the EWFr model with those fits of the KwFr, EFr, BFr, GEFr, Fr, and TFr distributions. Note that the EWFr model has the lowest values of the statistics. Therefore, the EWFr can be chosen as the best fitted model to the carbon fibre data.

The histogram, the estimated densities, cdfs, and QQ plots for cancer data are displayed in Figs. 6 and 7. Similarly, the histogram, the estimated densities, cdfs, and QQ plots for carbon fibre data are displayed in Figs. 8 and 9. These plots also provide a visual evidence that the EWLi model fits better to the cancer data than its sub-models and the EWFr model fits better to the carbon fibre data than its sub-models.

Table 7. MLEs and their standard errors (in parentheses) for carbon fibre data

Model	Estimates				
EWFr	$\hat{\alpha} = 0.413$ (1.295)	$\hat{\beta} = 0.1842$ (0.177)	$\hat{\nu} = 1.2991$ (0.557)	$\hat{a} = 1.3401$ (1.225)	$\hat{b} = 9.0999$ (8.270)
KwFr	$\hat{\alpha} = 2.0556$ (0.071)	$\hat{\beta} = 0.4654$ (0.00701)	$\hat{a} = 6.2815$ (0.063)	$\hat{b} = 224.18$ (0.164)	
BFr	$\hat{\alpha} = 1.6097$ (2.498)	$\hat{\beta} = 0.4046$ (0.108)	$\hat{a} = 22.0143$ (21.432)	$\hat{b} = 29.7617$ (17.479)	
GEFr	$\hat{\alpha} = 1.3692$ (2.017)	$\hat{\beta} = 0.4776$ (0.133)	$\hat{a} = 27.6452$ (14.136)	$\hat{b} = 17.4581$ (14.818)	
EFr	$\hat{\alpha} = 69.1489$ (57.349)	$\hat{\beta} = 0.5019$ (0.080)	$\hat{a} = 145.3275$ (122.924)		
TFr	$\hat{\alpha} = 1.9315$ (0.097)	$\hat{\beta} = 1.7435$ (0.076)	$\hat{a} = 0.0819$ (0.198)		
Fr	$\hat{\alpha} = 1.8705$ (0.112)	$\hat{\beta} = 1.7766$ (0.113)			

Table 8. Goodness-of-fit statistics for breaking stress of carbon fibre data

Model	$-2\widehat{\ell}$	AIC	BIC	HQIC	CAIC	W^*	A^*
EWFr	286.423	296.423	309.449	301.695	297.061	0.05624	0.34423
KwFr	289.059	297.095	307.479	301.276	297.48	0.09585	0.51495
EFr	289.697	295.697	303.513	298.861	295.947	0.10372	0.55798
BFr	303.133	311.133	321.553	315.35	311.554	0.25137	1.39536
GEFr	303.96	311.96	332.381	316.178	312.381	0.25872	1.43853
Fr	344.308	348.308	353.519	350.417	348.432	0.54849	3.13643
TFr	344.475	350.475	358.29	353.638	350.725	0.55598	3.17823

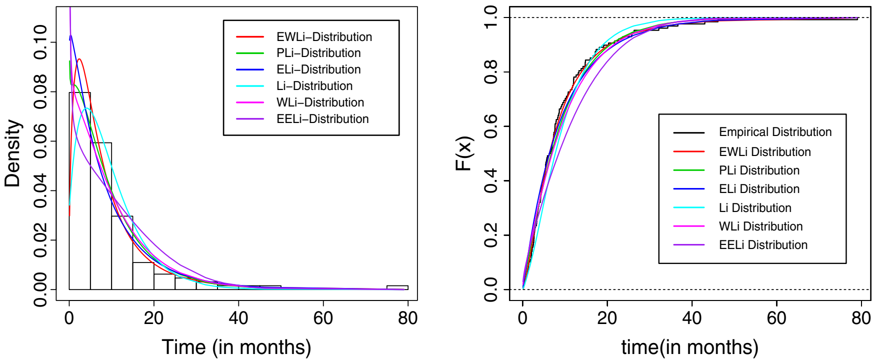

Figure 6. Estimated pdfs and cdfs of the EWLi model and its sub-models for the cancer data

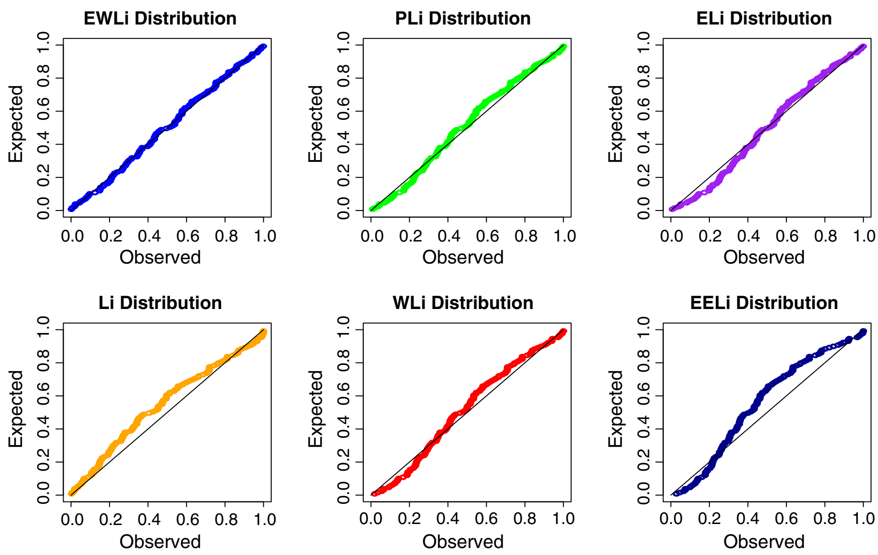

Figure 7. QQ plots of the EWLi model and its sub-models for the cancer data

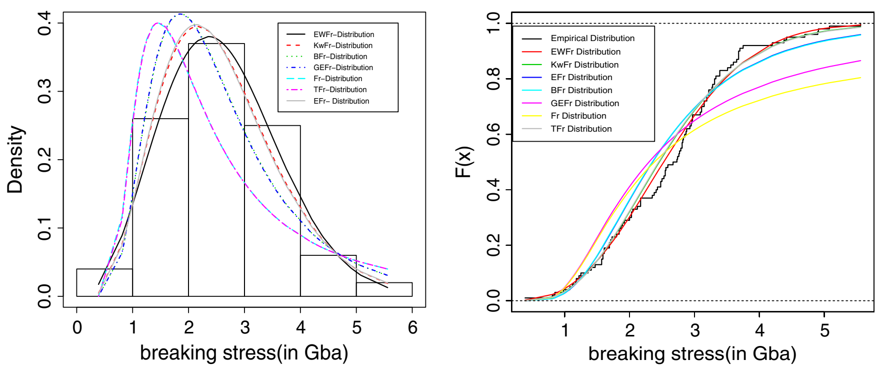

Figure 8. Estimated pdfs and cdfs of the EWFr model and other distributions for the carbon fibre data

11. Concluding Remarks

In many applied areas, there is a clear need for extended forms of the classical models, i.e., new distributions which are more flexible to capture skewness and kurtosis behavior. Recent developments focus on new techniques by adding parameters to existing distributions for building classes of more flexible distributions. In this study, we present a new *exponentiated Weibull*-H(EW-H) family of distributions, which extends the Weibull-H class by adding one extra shape parameter. Many well-known models emerge as special cases of the EW-H family by choosing special parameter values. Some mathematical properties of the new family including explicit expressions for the ordinary

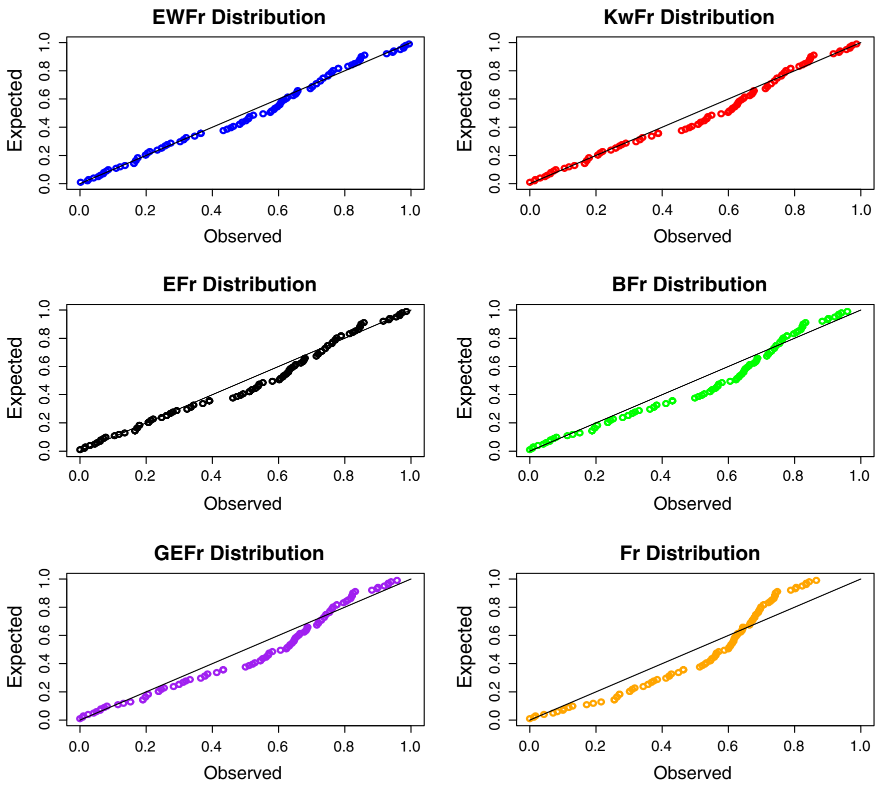

Figure 9. QQ plots of the EWFr model and other distributions for the carbon fibre data

and incomplete moments, quantile and generating functions, mean deviations, entropies, order statistics, and probability weighted moments are provided. The model parameters are estimated by maximum-likelihood and the observed information matrix is determined. We perform a Monte Carlo simulation study to assess the finite sample behavior of the maximum-likelihood estimators. We prove empirically by means of two real data sets that some special models of the EW-H family can give better fits than other models generated from well-known families.

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